

## ON CORRESPONDENCES OF A K3 SURFACE WITH ITSELF. II

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*To 60th Birthday of Igor Dolgachev*

ABSTRACT. Let  $X$  be a K3 surface with a polarization  $H$  of degree  $H^2 = 2rs$ ,  $r, s \geq 1$ , and the isotropic Mukai vector  $v = (r, H, s)$  is primitive. The moduli space of sheaves over  $X$  with the Mukai vector  $v = (r, H, s)$  is again a K3 surface,  $Y$ .

We prove that  $Y \cong X$  if the Picard lattice  $N(X)$  has an element  $h_1$  with  $(h_1)^2 = f(v)$  and minor additional congruence conditions modulo  $N_i(v)$ . All these conditions are exactly written, very efficient, and they are necessary if  $X$  is general with  $\text{rk } N(X) \leq 2$ .

Existence of such kind a criterion is very surprising, and it also gives some geometric interpretation of elements in  $N(X)$  with a negative square. Moreover, we describe all irreducible divisorial conditions on moduli of  $(X, H)$  which imply  $Y \cong X$ , and we prove that their number is always infinite.

Thus, we treat in general problems considered in [MN1], [MN2] and [N4], where the additional condition  $H \cdot N(X) = \mathbb{Z}$  had been imposed.

## 0. INTRODUCTION

Let  $X$  be a K3 surface with a polarization  $H$  of degree  $H^2 = 2rs$  where  $r, s \in \mathbb{N}$ . Assume that the isotropic Mukai vector  $v = (r, H, s)$  is primitive.

Let  $Y$  be the moduli space of sheaves (coherent and semi-stable with respect to  $H$ ) over  $X$  with the isotropic Mukai vector  $v = (r, H, s)$ . The  $Y$  (or, in special cases, its minimal resolution of singularities which we denote by the same letter  $Y$ ) is again a K3 surface which is equipped with a natural *nef* element  $h$  with  $h^2 = 2ab$  where we denote  $c = \text{g.c.d}(r, s)$  and  $a = r/c$ ,  $b = s/c$  (see Sect. 2.1 below). The surface  $Y$  is isogenous to  $X$  in the sense of Mukai. The second Chern class of the corresponding quasi-universal sheave gives then a 2-dimensional algebraic cycle  $Z \subset X \times Y$  and an algebraic correspondence between  $X$  and  $Y$ . See Mukai [Mu1]—[Mu5] and also Abe [A] about these results.

Let  $H$  be divisible by  $d \in \mathbb{N}$  where  $\tilde{H} = H/d$  is primitive in the Picard lattice  $N(X)$  of  $X$ . Primitivity of  $v = (r, H, s)$  means that  $\text{g.c.d}(r, d, s) = \text{g.c.d}(c, d) = 1$ . We have  $d^2 | ab$ . Let  $\gamma = \gamma(\tilde{H})$  is defined by  $\tilde{H} \cdot N(X) = \gamma\mathbb{Z}$ , i.e.  $H \cdot N(X) = \gamma d$ . Clearly,  $\gamma | (2rs/d^2) = \tilde{H}^2$ .

We denote

$$n(v) = \text{g.c.d}(r, s, d\gamma).$$

By Mukai, [Mu2], [Mu3],  $T(X) \subset T(Y)$ , and  $n(v) = [T(Y) : T(X)]$  where  $T(X)$  and  $T(Y)$  are transcendental lattices of  $X$  and  $Y$ . We assume that

$$n(v) = \text{g.c.d}(r, s, d\gamma) = \text{g.c.d}(c, d\gamma) = 1. \quad (0.1)$$

Since  $\text{g.c.d}(c, d) = 1$ , this is equivalent to  $\text{g.c.d}(c, \gamma) = 1$ . By Mukai [Mu2], [Mu3], the transcendental periods  $(T(X), H^{2,0}(X))$  and  $(T(Y), H^{2,0}(Y))$  are isomorphic in this case. We can expect that sometimes the surfaces  $X$  and  $Y$  are also isomorphic, and we then get a cycle  $Z \subset X \times X$ , and a correspondence of  $X$  with itself. Thus, an interesting for us question is

**Question 1.** *When is  $Y$  isomorphic to  $X$ ?*

We want to answer this question in terms of Picard lattices  $N(X)$  and  $N(Y)$  of  $X$  and  $Y$ . Then our question can be reformulated as follows:

**Question 2.** *Assume that  $N$  is a hyperbolic lattice,  $H_1 \in N$  an element with square  $2rs$ . What are conditions on  $N$  and  $H_1$  such that for any K3 surface  $X$  with Picard lattice  $N(X)$  and  $s$  polarization  $H \in N(X)$  the corresponding K3 surface  $Y$  is isomorphic to  $X$ , if the pairs  $(N(X), H)$  and  $(N, H_1)$  are isomorphic as abstract lattices with fixed elements?*

*In other words, what are conditions on  $(N(X), H)$  as an abstract lattice with an element  $H$  which are sufficient for  $Y$  to be isomorphic to  $X$ , and they are necessary, if  $X$  is a general K3 surface with the Picard lattice  $N(X)$ ?*

We answered this question in [MN1], [MN2] and [N4] under the condition  $d = \gamma = 1$  (equivalently,  $H \cdot N(X) = \mathbb{Z}$ ): in [MN1] for  $r = s = 2$ ; in [MN2] for  $r = s$ ; in [N4] for arbitrary  $r$  and  $s$ .

The main surprising result of [MN1], [MN2] and [N4] was that  $Y \cong X$  if the Picard lattice  $N(X)$  has an element  $h_1$  with some prescribed square  $(h_1)^2$  and some minor additional conditions. Moreover, these conditions are necessary to have  $Y \cong X$  for a general K3 surface  $X$  with  $\rho(X) = \text{rk } N(X) = 2$ . Thus, sometimes, elements of Picard lattice  $N(X)$  deliver important 2-dimensional algebraic cycles on  $X \times X$ . Moreover, here  $(h_1)^2$  can be negative, and this gives geometric meaning for elements of the Picard lattice with negative square (it is well-known only for  $h_1^2 = -2$ ; then  $\pm h_1$  is effective).

Here we prove similar results in general.

We assume (0.1). Then  $d^2 | ab$  and  $\gamma | 2ab/d^2$ . Moreover,  $\text{g.c.d}(a, b) = 1$ . We have  $d = d_a d_b$  where  $d_a = \text{g.c.d}(d, a)$  and  $d_b = \text{g.c.d}(d, b)$ . We define

$$a_1 = \frac{a}{d_a^2}, \quad b_1 = \frac{b}{d_b^2}.$$

We have  $\gamma = \gamma_2 \gamma_a \gamma_b$  where  $\gamma_a = \text{g.c.d}(a_1, \gamma)$ ,  $\gamma_b = \text{g.c.d}(b_1, \gamma)$ ,  $\gamma_2 = \gamma / (\gamma_a \gamma_b) | 2$ . We define

$$a_2 = \frac{a_1}{\gamma_a}, \quad b_2 = \frac{b_1}{\gamma_b}, \quad e_2 = \frac{2}{\gamma_2}.$$

Due to Mukai [Mu3] (see also Examples 2.3.4 and 2.3.5 below), one has the following result for  $\rho(X) = 1$ . *If  $\rho(X) = 1$  and  $X$  is a general K3 surface with its Picard lattice (i. e.  $\text{Aut}(T(X), H^{2,0}(X)) = \pm 1$ ), then  $Y \cong X$  if and only if  $c = 1$  and either  $a_1 = 1$  or  $b_1 = 1$ . In particular (e. g. see Lemma 2.1.1 below), for a primitive Mukai vector  $(r, H, s)$  one always has  $Y \cong X$  if and only if  $c = 1$  and either  $a_1 = 1$  or  $b_1 = 1$ .*

We prove (see Theorem 4.4) the following our main result for  $\rho(X) = 2$ . We denote as  $\mathbb{Z}f(\tilde{H})$  the orthogonal complement to  $\tilde{H}$  in the 2-dimensional lattice  $N$  and use the invariants  $\gamma, \delta \in \mathbb{N}$  and  $\mu \in (\mathbb{Z}/(2a_1b_1c^2/\gamma))^*$  of the pair  $\tilde{H} \in N$  (see Proposition 3.1.1). Here  $\tilde{H} \cdot N = \gamma\mathbb{Z}$ ,  $\det N = -\gamma\delta$ ,  $N = [\tilde{H}, f(\tilde{H}), (\mu\tilde{H} + f(\tilde{H}))/ (2a_1b_1c^2/\gamma)]$ . One always has  $\delta \equiv \gamma\mu^2 \pmod{4a_1b_1c^2/\gamma}$ . Moreover, below  $\tilde{h}_1 = (p_1\tilde{H} + q_1f(\tilde{H}))/((2/\gamma_2)(a_1/\gamma_a)c)$  for  $a$ -series, and  $\tilde{h}_1 = (p_1\tilde{H} + q_1f(\tilde{H}))/((2/\gamma_2)(b_1/\gamma_b)c)$  for  $b$ -series. Also we denote by  $n^{(l)}$  the  $l$ -component of a natural number  $n$  for a prime  $l$ , i. e.  $n^{(l)} = l^k | n$  and  $\text{g.c.d}(n, n/n^{(l)}) = 1$ .

**Theorem 0.1.** *Let  $X$  be a K3 surface and  $H$  a polarization of  $X$  of degree  $H^2 = 2rs$  where  $r, s \in \mathbb{N}$ . Assume that the Mukai vector  $(r, H, s)$  is primitive. Let  $Y$  be the moduli space of sheaves on  $X$  with the isotropic Mukai vector  $v = (r, H, s)$ . Let  $\tilde{H} = H/d$  be the corresponding primitive polarization.*

*We have  $Y \cong X$  if there exists  $\tilde{h}_1 \in N(X)$  such that  $\tilde{H}, \tilde{h}_1$  belong to a 2-dimensional primitive sublattice  $N \subset N(X)$  such that  $\tilde{H} \cdot N = \gamma\mathbb{Z}$ ,  $\gamma > 0$ , and*

$$\text{g.c.d}(c, d\gamma) = 1,$$

*moreover, for one of  $\epsilon = \pm 1$  the element  $\tilde{h}_1$  belongs to the  $a$ -series or to the  $b$ -series described below:*

*$\tilde{h}_1$  belongs to the  $a$ -series if*

$$\tilde{h}_1^2 = \epsilon 2b_1c \text{ and } \tilde{H} \cdot \tilde{h}_1 \equiv 0 \pmod{\gamma(b_1/\gamma_b)c},$$

$$\tilde{H} \cdot \tilde{h}_1 \not\equiv 0 \pmod{\gamma(b_1/\gamma_b)cl_1}, \quad \tilde{h}_1/l_2 \notin N(X)$$

*for any prime  $l_1$  such that  $l_1^2 | a_1$  and  $\text{g.c.d}(l_1, \gamma) = 1$ , and any prime  $l_2$  such that  $l_2^2 | b_1$  and  $\text{g.c.d}(l_2, \gamma) = 1$ , and*

$$p_1 = \frac{\tilde{H} \cdot \tilde{h}_1}{\gamma(b_1/\gamma_b)c}, \quad q_1 = -\frac{f(\tilde{H}) \cdot \tilde{h}_1}{\delta(b_1/\gamma_b)c}$$

satisfy the singular condition (AS) of  $a$ -series:

- if odd prime  $l|\gamma$  and  $l^2|a_1$ , then  $q_1 \not\equiv 0 \pmod{l}$  and  
either  $\delta \not\equiv 0 \pmod{l}$  or  $(\delta - \gamma\mu^2) \not\equiv 0 \pmod{(a_1^{(l)}/\gamma_a^{(l)})l}$ ;
- if odd prime  $l|\gamma$  and  $l|b_1$ , then  $q_1 \equiv 0 \pmod{\gamma_b^{(l)}}$ ;
- if odd prime  $l|\gamma$  and  $l^2|b_1$ , then  $p_1 \not\equiv 0 \pmod{l}$ ;
- if  $2|\gamma$ ,  $\gamma_2 = 1$  and  $2|a_1$ , then  $p_1 \equiv 1 \pmod{2}$ ;
- if  $2|\gamma$ ,  $\gamma_2 = 1$  and  $4|a_1$ , then  $\delta - \gamma\mu^2 \not\equiv 0 \pmod{(8a_1b_1c^2/\gamma)}$ ;
- if  $2|\gamma$ ,  $\gamma_2 = 1$ , and  $2|b_1$ , then  $p_1 - \mu q_1 \not\equiv 0 \pmod{4}$  and  $q_1 \equiv 0 \pmod{\gamma_b^{(2)}}$ ;
- if  $2|\gamma$ ,  $\gamma_2 = 2$  and  $2|b_1$ , then  $p_1 \equiv 1 \pmod{2}$  and  $q_1 \equiv 0 \pmod{\gamma^{(2)}/2}$ .

$\tilde{h}_1$  belongs to the  $b$ -series if

$$\tilde{h}_1^2 = \epsilon 2a_1c \text{ and } \tilde{H} \cdot \tilde{h}_1 \equiv 0 \pmod{\gamma(a_1/\gamma_a)c},$$

$$\tilde{H} \cdot \tilde{h}_1 \not\equiv 0 \pmod{\gamma(a_1/\gamma_a)cl_1}, \quad \tilde{h}_1/l_2 \notin N(X)$$

for any prime  $l_1$  such that  $l_1^2|b_1$  and  $\text{g.c.d}(l_1, \gamma) = 1$  and any prime  $l_2$  such that  $l_2^2|a_1$  and  $\text{g.c.d}(l_2, \gamma) = 1$ , and

$$p_1 = \frac{\tilde{H} \cdot \tilde{h}_1}{\gamma(a_1/\gamma_a)c}, \quad q_1 = -\frac{f(\tilde{H}) \cdot \tilde{h}_1}{\delta(a_1/\gamma_a)c}$$

satisfy the singular condition (BS) of  $b$ -series:

- if odd prime  $l|\gamma$  and  $l|a_1$ , then  $q_1 \equiv 0 \pmod{\gamma_a^{(l)}}$ ;
- if odd prime  $l|\gamma$  and  $l^2|a_1$ , then  $p_1 \not\equiv 0 \pmod{l}$ ;
- if odd prime  $l|\gamma$  and  $l^2|b_1$ , then  $q_1 \not\equiv 0 \pmod{l}$  and  
either  $\delta \not\equiv 0 \pmod{l}$  or  $(\delta - \gamma\mu^2) \not\equiv 0 \pmod{(b_1^{(l)}/\gamma_b^{(l)})l}$ ;
- if  $2|\gamma$ ,  $\gamma_2 = 1$ , and  $2|a_1$ , then  $p_1 - \mu q_1 \not\equiv 0 \pmod{4}$  and  $q_1 \equiv 0 \pmod{\gamma_a^{(2)}}$ ;
- if  $2|\gamma$ ,  $\gamma_2 = 1$  and  $2|b_1$ , then  $p_1 \equiv 1 \pmod{2}$ ;
- if  $2|\gamma$ ,  $\gamma_2 = 1$  and  $4|b_1$ , then  $\delta - \gamma\mu^2 \not\equiv 0 \pmod{(8a_1b_1c^2/\gamma)}$ ;
- if  $2|\gamma$ ,  $\gamma_2 = 2$  and  $2|a_1$ , then  $p_1 \equiv 1 \pmod{2}$  and  $q_1 \equiv 0 \pmod{\gamma^{(2)}/2}$ .

Moreover, one has formulae (4.23) and (4.24) in terms of  $X$  for the canonical primitive nef element  $\tilde{h}$  of  $Y$  defined by  $(-a, 0, b) \pmod{\mathbb{Z}v}$ .

These conditions are necessary to have  $Y \cong X$  if  $\rho(X) \leq 2$  and  $X$  is a general K3 surface with its Picard lattice, i. e. the automorphism group of the transcendental periods  $(T(X), H^{2,0}(X))$  is  $\pm 1$ .

As concrete examples, in Sect. 6 we specialize the theorem for  $\gamma = 1$  and  $\gamma = 2$ . The same can be done for any  $\gamma$ .

For the Mukai case when  $c = 1$  and either  $a_1 = 1$  or  $b_1 = 1$ , one satisfies conditions of Theorem 0.1 for  $\tilde{h}_1 = \tilde{H}$ .

It seems many (if not all) known examples when  $\rho(X) \geq 2$  and  $Y \cong X$  follow from this Theorem. E.g. see [C], [T1]—[T3] and [V].

Like in [MN1], [MN2] and [N4] we also describe all irreducible divisorial conditions on moduli of polarized K3 surfaces  $(X, H)$  which imply  $\tilde{H} \cdot N(X) = \gamma\mathbb{Z}$  and  $Y \cong X$ . We show that they are labelled by pairs  $(\pm\mu, \delta)$  where  $\pm\mu \in (\mathbb{Z}/(2a_1b_1c^2/\gamma))^*$ ,  $\delta \in \mathbb{N}$  and  $\delta \equiv \mu^2\gamma \pmod{4a_1b_1c^2/\gamma}$ , moreover the pair belongs to the  $a$ -series or to the  $b$ -series. It belongs to the  $a$ -series if at least for one  $\epsilon = \pm 1$  the equation

$$\gamma p_1^2 - \delta q_1^2 = \epsilon 2(2/\gamma_2)(a_1/\gamma_a)\gamma_b c$$

has an integral solution  $(p_1, q_1)$  where  $(p_1, q_1)$  satisfy conditions (A) of  $a$ -series (3.3.54)—(3.3.57). Similarly one can consider  $b$ -series changing  $a$  and  $b$  places. See Sect. 4.

In Sect. 5, as an application, we prove that the number of the irreducible divisorial conditions is infinite if non-empty. If  $\gamma = 1$ , the same considerations as for  $d = \gamma = 1$  in [N4] show that for any type of a primitive isotropic Mukai vector  $(r, H, s)$  the number of divisorial conditions on moduli of K3 which imply that  $Y \cong X$  and  $\gamma = 1$  is always non-empty and infinite. In particular, for any type of a primitive isotropic Mukai vector the number of divisorial conditions on moduli of K3 which imply that  $Y \cong X$  is always non-empty and infinite.

This paper generalizes to the general case results of [N4] (see also [MN1] and [MN2]) where a particular case  $d = \gamma = 1$  had been considered.

As in [MN1], [MN2] and [N4], the fundamental tools to get the results above is the Global Torelli Theorem for K3 surfaces proved by Piatetsky-Shapiro and Shafarevich in [PS], and results of Mukai [Mu2], [Mu3]. By results of [Mu2], [Mu3], we can calculate periods of  $Y$  using periods of  $X$ ; comparing the periods, by the Global Torelli Theorem for K3 surfaces [PS], we can find out if  $Y$  is isomorphic to  $X$ .

These paper treats in general problems considered in [MN1], [MN2] and [N4] where the additional condition  $\gamma = d = 1$  had been imposed. It makes results of this paper much more complicated. For instance, in these paper we don't consider the question of non-emptiness of the divisorial conditions on moduli for  $\gamma > 1$ . It is more difficult in the general setting of this paper. We hope to consider this problem later.

## 1. PRELIMINARY NOTATIONS AND RESULTS ABOUT LATTICES AND K3 SURFACES

**1.1. Some notations about lattices.** We use notations and terminology from [N2] about lattices, their discriminant groups and forms. A *lattice*  $L$  is a non-degenerate integral symmetric bilinear form. I. e.  $L$  is a free  $\mathbb{Z}$ -module equipped

with a symmetric pairing  $x \cdot y \in \mathbb{Z}$  for  $x, y \in L$ , and this pairing should be non-degenerate. We denote  $x^2 = x \cdot x$ . The *signature* of  $L$  is the signature of the corresponding real form  $L \otimes \mathbb{R}$ . The lattice  $L$  is called *even* if  $x^2$  is even for any  $x \in L$ . Otherwise,  $L$  is called *odd*. The *determinant* of  $L$  is defined to be  $\det L = \det(e_i \cdot e_j)$  where  $\{e_i\}$  is some basis of  $L$ . The lattice  $L$  is *unimodular* if  $\det L = \pm 1$ . The *dual lattice* of  $L$  is  $L^* = \text{Hom}(L, \mathbb{Z}) \subset L \otimes \mathbb{Q}$ . The *discriminant group* of  $L$  is  $A_L = L^*/L$ . It has the order  $|\det L|$ . The group  $A_L$  is equipped with the *discriminant bilinear form*  $b_L : A_L \times A_L \rightarrow \mathbb{Q}/\mathbb{Z}$  and the *discriminant quadratic form*  $q_L : A_L \rightarrow \mathbb{Q}/2\mathbb{Z}$  if  $L$  is even. To get this forms, one should extend the form of  $L$  to the form on the dual lattice  $L^*$  with values in  $\mathbb{Q}$ .

For  $x \in L$ , we shall consider the invariant  $\gamma(x) \geq 0$  where

$$x \cdot L = \gamma(x)\mathbb{Z}. \quad (1.1.1)$$

Clearly,  $\gamma(x)|x^2$  if  $x \neq 0$ .

We denote by  $L(k)$  the lattice obtained from a lattice  $L$  by multiplication of the form of  $L$  by  $k \in \mathbb{Q}$ . The orthogonal sum of lattices  $L_1$  and  $L_2$  is denoted by  $L_1 \oplus L_2$ . For a symmetric integral matrix  $A$ , we denote by  $\langle A \rangle$  a lattice which is given by the matrix  $A$  in some bases. We denote

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (1.1.2)$$

Any even unimodular lattice of the signature  $(1, 1)$  is isomorphic to  $U$ .

An embedding  $L_1 \subset L_2$  of lattices is called *primitive* if  $L_2/L_1$  has no torsion. We denote by  $O(L)$ ,  $O(b_L)$  and  $O(q_L)$  the automorphism groups of the corresponding forms. Any  $\delta \in L$  with  $\delta^2 = -2$  defines a reflection  $s_\delta \in O(L)$  which is given by the formula

$$x \rightarrow x + (x \cdot \delta)\delta,$$

$x \in L$ . All such reflections generate the *2-reflection group*  $W^{(-2)}(L) \subset O(L)$ .

**1.2. Some notations about K3 surfaces.** Here we remind some basic notions and results about K3 surfaces, e. g. see [PS], [S-D], [Sh]. A K3 surface  $X$  is a non-singular projective algebraic surface over  $\mathbb{C}$  such that its canonical class  $K_X$  is zero and the irregularity  $q_X = 0$ . We denote by  $N(X)$  the *Picard lattice* of  $X$  which is a hyperbolic lattice with the intersection pairing  $x \cdot y$  for  $x, y \in N(X)$ . Since the canonical class  $K_X = 0$ , the space  $H^{2,0}(X)$  of 2-dimensional holomorphic differential forms on  $X$  has dimension one over  $\mathbb{C}$ , and

$$N(X) = \{x \in H^2(X, \mathbb{Z}) \mid x \cdot H^{2,0}(X) = 0\} \quad (1.2.1)$$

where  $H^2(X, \mathbb{Z})$  with the intersection pairing is a 22-dimensional even unimodular lattice of signature  $(3, 19)$ . The orthogonal lattice  $T(X)$  to  $N(X)$  in  $H^2(X, \mathbb{Z})$  is called the *transcendental lattice* of  $X$ . We have  $H^{2,0}(X) \subset T(X) \otimes \mathbb{C}$ . The pair

$(T(X), H^{2,0}(X))$  is called the *transcendental periods of  $X$* . The *Picard number* of  $X$  is  $\rho(X) = \text{rk } N(X)$ . A non-zero element  $x \in N(X) \otimes \mathbb{R}$  is called *nef* if  $x \neq 0$  and  $x \cdot C \geq 0$  for any effective curve  $C \subset X$ . It is known that an element  $x \in N(X)$  is ample (i. e. it defines a polarization) if  $x^2 > 0$ ,  $x$  is *nef*, and the orthogonal complement  $x^\perp$  to  $x$  in  $N(X)$  has no elements with square  $-2$ . For any non-zero element  $x \in N(X)$  with  $x^2 \geq 0$ , there exists a reflection  $w \in W^{(-2)}(N(X))$  such that the element  $\pm w(x)$  is nef; it then is ample if  $x^2 > 0$  and  $x^\perp$  had no elements with square  $-2$  in  $N(X)$ . The *nef* element  $\pm w(x)$  is defined canonically by  $x$ . It is called *the canonical nef element of  $x$* .

We denote by  $V^+(X)$  the light cone of  $X$ , which is the half-cone of

$$V(X) = \{x \in N(X) \otimes \mathbb{R} \mid x^2 > 0\} \quad (1.2.2)$$

containing a polarization of  $X$ . In particular, all *nef* elements  $x$  of  $X$  belong to  $\overline{V^+(X)}$ : one has  $x \cdot V^+(X) > 0$  for them.

The reflection group  $W^{(-2)}(N(X))$  acts in  $V^+(X)$  discretely, and its fundamental chamber is the closure  $\overline{\mathcal{K}(X)}$  of the Kähler cone  $\mathcal{K}(X)$  of  $X$ . It is the same as the set of all *nef* elements of  $X$ . Its faces are orthogonal to the set  $\text{Exc}(X)$  of all exceptional curves  $r$  on  $X$  which are non-singular rational curves  $r$  on  $X$  with  $r^2 = -2$ . Thus, we have

$$\overline{\mathcal{K}(X)} = \{0 \neq x \in \overline{V^+(X)} \mid x \cdot \text{Exc}(X) \geq 0\}. \quad (1.2.3)$$

## 2. CONDITION OF $Y \cong X$ FOR A GENERAL K3 SURFACE $X$ WITH A GIVEN PICARD LATTICE

**2.1. The correspondence.** Let  $X$  be a smooth complex projective K3 surface with a polarization  $H$  of degree  $2rs$  where  $r, s \in \mathbb{N}$ .

Assume that  $H$  is divisible by  $d \in \mathbb{N}$  and  $\tilde{H} = H/d$  is primitive in  $N(X)$ . Then  $\tilde{H}^2 = 2rs/d^2$  and  $d^2 \mid rs$ . We denote

$$c = \text{g.c.d.}(r, s), \quad a = r/c, \quad b = s/c. \quad (2.1.1)$$

We assume that the Mukai vector  $(r, H, s)$  is primitive, i. e.

$$\text{g.c.d.}(r, s, d) = \text{g.c.d.}(c, d) = 1. \quad (2.1.2)$$

Let  $Y$  be the moduli space of sheaves  $\mathcal{E}$  (coherent and semi-stable with respect to  $H$ ) on  $X$  with the primitive isotropic Mukai vector  $v = (r, H, s)$ . Then  $\text{rk } \mathcal{E} = r$ ,  $\chi(\mathcal{E}) = r + s$  and  $c_1(\mathcal{E}) = H$ . The  $Y$  (or, in special cases, its minimal resolution of singularities which we denote by the same letter  $Y$ ) is again a K3 surface. See [Mu1]—[Mu5] and also [A] about these results.

Let

$$H^*(X, \mathbb{Z}) = H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z}) \quad (2.1.3)$$

be the full cohomology lattice of  $X$  equipped with the Mukai pairing

$$(u, v) = -(u_0 \cdot v_2 + u_2 \cdot v_0) + u_1 \cdot v_1 \quad (2.1.4)$$

for  $u_0, v_0 \in H^0(X, \mathbb{Z})$ ,  $u_1, v_1 \in H^2(X, \mathbb{Z})$ ,  $u_2, v_2 \in H^4(X, \mathbb{Z})$ . We naturally identify  $H^0(X, \mathbb{Z})$  and  $H^4(X, \mathbb{Z})$  with  $\mathbb{Z}$ . Then the Mukai pairing is

$$(u, v) = -(u_0 v_2 + u_2 v_0) + u_1 \cdot v_1. \quad (2.1.5)$$

The element

$$v = (r, H, s) = (r, H, \chi - r) \in H^*(X, \mathbb{Z}) \quad (2.1.6)$$

is isotropic, i.e.  $v^2 = 0$ , since  $H^2 = 2rs$ . In this case (for a primitive  $v$ ), Mukai [Mu2]—[Mu5] (see also Abe [A]) showed that  $Y$  is a K3 surface, and one has the natural identification

$$H^2(Y, \mathbb{Z}) \cong (v^\perp / \mathbb{Z}v) \quad (2.1.7)$$

which also gives the isomorphism of the Hodge structures of  $X$  and  $Y$ , i. e.  $H^{2,0}(Y)$  will be identified with the image of  $H^{2,0}(X)$ . The  $Y$  has the canonical *nef* element  $h$  defined by  $(-a, 0, b) \bmod \mathbb{Z}v$  with  $h^2 = 2ab$  (see Sect. 1.2).

In particular, (2.1.7) gives the embedding

$$T(X) \subset T(Y) \quad (2.1.8)$$

of the transcendental lattices of the index

$$[T(Y) : T(X)] = n(v) = \min |v \cdot x| \quad (2.1.9)$$

where  $x \in H^0(X, \mathbb{Z}) \oplus N(X) \oplus H^4(X, \mathbb{Z})$  and  $v \cdot x \neq 0$  (see [Mu2], [Mu3]). In this paper, we are interested in the case when  $Y \cong X$ . By (2.1.9), it may happen if  $n(v) = 1$  only.

We can introduce the invariant  $\gamma = \gamma(\tilde{H}) \in \mathbb{N}$  which is defined by

$$\tilde{H} \cdot N(X) = \gamma \mathbb{Z}, \quad (2.1.10)$$

equivalently,  $H \cdot N(X) = \gamma d \mathbb{Z}$ . Clearly,  $\gamma |2rs/d^2 = \tilde{H}^2$ , and

$$n(v) = \text{g.c.d}(r, s, \gamma d) = \text{g.c.d}(c, \gamma d). \quad (2.1.11)$$

Thus,  $n(v) = 1$ , and it is possible to have  $Y \cong X$  only if

$$\text{g.c.d}(r, s, \gamma d) = \text{g.c.d}(c, \gamma) = \text{g.c.d}(c, d) = 1. \quad (2.1.12)$$

This is exactly the case when, according to Mukai, the transcendental periods

$$(T(X), H^{2,0}(X)) \cong (T(Y), H^{2,0}(Y)) \quad (2.1.13)$$

are isomorphic.

From (2.1.7), we obtain the following *specialization principle*.

We say that a K3 surface  $X$  is *general* (for its Picard lattice) if the automorphism group of the transcendental periods  $\text{Aut}(T(X), H^{2,0}(X)) = \pm 1$ . We have



**Lemma 2.1.1.** *(The specialization principle.) Assume that for a general K3 surface  $X$  with  $N = N(X)$  and a primitive isotropic Mukai vector  $v = (r, H, s)$  where  $H \in N$  is a polarization of  $X$ , one has  $Y \cong X$*

*Then the same is valid for any K3 surface  $X'$  such that  $H \in N \subset N(X')$  if  $N \subset N(X')$  is a primitive sublattice in  $N(X')$  and  $H$  is a polarization of  $X$ .*

*Proof.* Since  $Y \cong X$  and  $X$  is general, there exist only two isomorphisms of transcendental periods  $(T(X), H^{2,0}(X)) \cong (T(Y), H^{2,0}(Y))$  which are  $\pm 1$  of the identification of the transcendental periods  $(T(X), H^{2,0}(X)) = (T(Y), H^{2,0}(Y))$  which is defined by Mukai identification (2.1.7). Since  $Y \cong X$ , there exists an extension of this identification to the isomorphism  $H^2(X, \mathbb{Z}) \cong H^2(Y, \mathbb{Z})$  of the cohomology lattices.

Now assume that  $H \in N \subset N(X')$ . Let  $Y'$  be the corresponding moduli space of sheaves on  $X'$  with the same Mukai vector  $v$ .

By local epimorphicity of the period map for K3, we can assume that  $N = N(X)$  is the Picard lattice of a K3 surface  $X$  with the polarization  $H$ ,  $X$  is general and the embedding  $N(X) \subset N(X')$  extends to the identification of the cohomology lattices  $H^2(X, \mathbb{Z}) = H^2(X', \mathbb{Z})$ . Then  $T(X) \supset T(X')$  is a primitive sublattice. By the Mukai identification (2.1.7), the identification (2.1.7)  $(T(X'), H^{2,0}(X')) = (T(Y'), H^{2,0}(Y'))$  is extending to the identification of the transcendental periods  $(T(X), H^{2,0}(X)) = (T(Y), H^{2,0}(Y))$  and to the identification of the cohomology lattices  $H^2(Y, \mathbb{Z}) = H^2(Y', \mathbb{Z})$ . Since  $X$  is general,  $Y \cong X$ , and the identification above of their transcendental periods extends to an isomorphism of the lattices  $H^2(X, \mathbb{Z}) \cong H^2(Y, \mathbb{Z})$ . This gives the isomorphism  $H^2(X', \mathbb{Z}) = H^2(X, \mathbb{Z}) \cong H^2(Y, \mathbb{Z}) = H^2(Y', \mathbb{Z})$  which extends the above isomorphism  $(T(X'), H^{2,0}(X')) \cong (T(Y'), H^{2,0}(Y'))$ .

By global Torelli Theorem for K3 surfaces [PS], this defines an isomorphism  $Y' \cong X'$ . This finishes the proof.

**2.2. The characteristic map of a primitive element of a lattice.** Let  $S$  be an even lattice and  $P \in S$  its primitive element with  $P^2 = 2m \neq 0$  and  $\gamma(P) = \gamma|2m$  (in  $S$ ), i. e.  $P \cdot S = \gamma\mathbb{Z}$ .

We want to calculate the discriminant quadratic form of  $S$ . Consider

$$K(P) = P_S^\perp \tag{2.2.1}$$

the orthogonal complement to  $P$  in  $S$ . Put  $P^* = P/2m$ . Then any element  $x \in S$  can be written as

$$x = n\gamma P^* + k^* \tag{2.2.2}$$

where  $n \in \mathbb{Z}$  and  $k^* \in K(P)^*$ , because

$$\mathbb{Z}P \oplus K(P) \subset S \subset S^* \subset \mathbb{Z}P^* \oplus K(P)^*$$

and  $P \cdot S = \gamma\mathbb{Z}$ . Since  $\gamma(P) = \gamma$ , the map  $n\gamma P^* + [P] \rightarrow k^* + K(P)$  gives an isomorphism of the groups  $\mathbb{Z}/\frac{2m}{\gamma} \cong [\gamma P^*]/[P] \cong [u^*(P) + K(P)]/K(P)$  where

$u^*(P) + K(P)$  has order  $2m/\gamma$  in  $A_{K(P)} = K(P)^*/K(P)$ . Clarify in [N2]. It follows,

$$S = [\mathbb{Z}P, K(P), \gamma P^* + u^*(P)]. \quad (2.2.3)$$

The element  $u^*(P)$  is defined canonically mod  $K(P)$  by the condition that  $\gamma P^* + u^* \in S$ . The element

$$u^*(P) + K(P) \in K(P)^*/K(P) \quad (2.2.4)$$

is called *the canonical element of P*. Since  $\gamma P^* + u^*(P)$  belongs to the even lattice  $S$ , it follows

$$(\gamma P^* + u^*(P))^2 = \frac{\gamma^2}{2m} + u^*(P)^2 \equiv 0 \pmod{2}. \quad (2.2.5)$$

For  $n \in \mathbb{Z}$  and  $k^* \in K(P)^*$ , we have  $x = nP^* + k^* \in S^*$  if and only if

$$(nP^* + k^*) \cdot (\gamma P^* + u^*(P)) = \frac{n\gamma}{2m} + k^* \cdot u^*(P) \in \mathbb{Z}.$$

It follows,

$$\begin{aligned} S^* &= \{nP^* + k^* \mid n \in \mathbb{Z}, k^* \in K(P)^*, n \equiv -\frac{2m}{\gamma} (k^* \cdot u^*(P)) \pmod{\frac{2m}{\gamma}}\} \subset \\ &\subset \mathbb{Z}P^* + K(P)^*. \end{aligned} \quad (2.2.6)$$

It gives the calculation of the discriminant group  $A_S = S^*/S$  where  $S$  is given by (2.2.3) and  $S^*$  is given by (2.2.6).

We define *the canonical submodule*  $\tilde{K}(P)^* \subset \mathbb{Z} \oplus K(P)^*$  by the condition

$$\tilde{K}(P)^* = \{n \in \mathbb{Z}, k^* \in K(P)^* \mid n \equiv -\frac{2m}{\gamma} (k^* \cdot u^*(P)) \pmod{\frac{2m}{\gamma}}\}. \quad (2.2.7)$$

Now we define the *characteristic map*

$$\kappa(P) : \tilde{K}(P)^* \rightarrow A_S, \quad (2.2.8)$$

by the condition

$$\kappa(P)(n, k^*) = nP^* + k^* \pmod{S}. \quad (2.2.9)$$

Obviously, the characteristic map is epimorphic. Its kernel is

$$\tilde{K}(P)_0^* = [(2m\mathbb{Z}, K), \mathbb{Z}(\gamma, u^*(P))] \cong S. \quad (2.2.10)$$

Thus, we correspond to a primitive  $P \in S$  with  $P^2 = 2m$  and  $\gamma(P) = \gamma$  the canonical triplet

$$(K(P), u^*(P) + K(P), \kappa(P)). \quad (2.2.11)$$

This triplet is important because of the trivial but very important for us

**Lemma 2.2.1.** *Let  $P_1 \in S$  and  $P_2 \in S$  are two primitive elements of an even lattice  $S$  with  $P_1^2 \neq 0$  and  $P_2^2 \neq 0$ .*

*There exists an automorphism  $f \in O(S)$  such that  $f(P_1) = P_2$  and  $f$  gives  $\pm 1$  on the discriminant group  $A_S = S^*/S$  if and only if  $P_1^2 = P_2^2$ ,  $\gamma(P_1) = \gamma(P_2)$ , and there exists an isomorphism of lattices  $\phi : K(P_1) \rightarrow K(P_2)$  such that  $\phi^*(u^*(P_2) + K(P_2)) = u^*(P_1) + K(P_1)$  and  $\tilde{\phi}^* = (id, \phi^*) : \tilde{K}(P_2)^* \rightarrow \tilde{K}(P_1)^*$  is  $\pm$  commuting with the characteristic maps  $\kappa(P_1)$  and  $\kappa(P_2)$ , i. e.*

$$\kappa(P_1)\tilde{\phi}^* = \pm \kappa(P_2).$$

*Proof.* Trivial.

We also mention that

$$\det(S) = \gamma^2 \det K(P)/2m. \quad (2.2.12)$$

because  $[S : \mathbb{Z}P \oplus K(P)] = 2m/\gamma$

**2.3. Relation between periods of  $X$  and  $Y$ .** Here we consider the case and notations of Sect. 2.1. Thus, for the primitive isotropic Mukai vector  $v = (r, H, s)$ ,  $r, s \geq 1$ ,  $H^2 = 2rs$ , we assume that for  $d \in \mathbb{N}$ , the element  $\tilde{H} = H/d$  is primitive in  $N(X)$ ,  $\gamma(\tilde{H}) = \gamma$  (in  $N(X)$ ). We remind that  $c = \text{g.c.d}(r, s)$ ,  $a = r/c$ ,  $b = s/c$ , and  $n(v) = \text{g.c.d}(r, s, d\gamma) = (c, d\gamma) = 1$ . It follows,  $d^2|ab$  and  $\gamma d^2|2ab$ . Thus, our data are defined by

$$a, b, c, d, \gamma \in \mathbb{N}, \text{ such that } (a, b) = (d, c) = (d, \gamma) = 1, d^2|ab, \gamma d^2|2ab, \quad (2.3.1)$$

and by a primitive polarization

$$\tilde{H} \in N(X) \text{ such that } \tilde{H}^2 = 2abc^2/d^2, \gamma(\tilde{H}) = \gamma. \quad (2.3.2)$$

Then, the Mukai vector  $v = (r, H, s) = (ac, d\tilde{H}, bc)$ .

Let us denote by  $e_1$  the canonical generator of  $H^0(X, \mathbb{Z})$  and by  $e_2$  the canonical generator of  $H^4(X, \mathbb{Z})$ . They generate the sublattice  $U$  in  $H^*(X, \mathbb{Z})$  with the Gram matrix  $U$ . Consider Mukai vector  $v = (re_1 + se_2 + H)$ . We have

$$N(Y) = v^\perp_{U \oplus N(X)} / \mathbb{Z}v. \quad (2.3.3)$$

Let us calculate  $N(Y)$ . Let  $K(H) = K(\tilde{H}) = (H)^\perp_{N(X)}$ . We denote  $H^* = H/(2rs) \in (\mathbb{Z}H)^* = \mathbb{Z}H^*$ , and  $\tilde{H}^* = \tilde{H}/(2rs/d^2) = dH^* \in (\mathbb{Z}\tilde{H})^* = \mathbb{Z}\tilde{H}^*$ . Then we have an embedding of lattices of finite index

$$\mathbb{Z}\tilde{H} \oplus K(H) \subset N(X) \subset N(X)^* \subset \mathbb{Z}\tilde{H}^* \oplus K(H)^* \quad (2.3.4)$$

We have the orthogonal decomposition up to finite index

$$U \oplus \mathbb{Z}\tilde{H} \oplus K(H) \subset U \oplus N(X) \subset U \oplus \mathbb{Z}\tilde{H}^* \oplus K(H)^*. \quad (2.3.5)$$

Let  $f = x_1e_1 + x_2e_2 + yH^* + z^* \in v_{U \oplus N(X)}^\perp$ ,  $z^* \in K(H)^*$ . Then  $-sx_1 - rx_2 + y = 0$  since  $f \in v^\perp$  and hence  $(f, v) = 0$ . Thus,  $y = (sx_1 + rx_2)$  and

$$f = x_1e_1 + x_2e_2 + (sx_1 + rx_2)H^* + z^*. \quad (2.3.6)$$

Here  $f \in U \oplus N(X)$  if and only if

$$x_1, x_2 \in \mathbb{Z}, \quad sx_1 + rx_2 \equiv 0 \pmod{d}, \quad \frac{sx_1 + rx_2}{d}\tilde{H}^* + z^* \in N(X). \quad (2.3.7)$$

Equivalently, by Sect. 2.2, we have

$$sx_1 + rx_2 \equiv 0 \pmod{d\gamma}, \quad z^* = \frac{sx_1 + rx_2}{d\gamma}u^*(H) \pmod{K(H)}. \quad (2.3.8)$$

We denote

$$h' = (-a, b) \oplus 0 \in U \oplus N(X).$$

Clearly,  $h' \in v^\perp$  and  $h = h' \pmod{\mathbb{Z}v} \in N(Y)$ . Thus, the orthogonal complement contains

$$[\mathbb{Z}v, \mathbb{Z}h', K(H)] \quad (2.3.9)$$

where  $h' = -ae_1 + be_2$ , and (2.3.9) is a sublattice of finite index in  $(v^\perp)_{U \oplus N(X)}$ . The generators  $v$ ,  $h'$  and generators of  $K(H)$  are free, and we can rewrite  $f$  above using these generators with rational coefficients. We have

$$e_1 = \frac{v - ch' - H}{2r}, \quad e_2 = \frac{v + ch' - H}{2s}. \quad (2.3.10)$$

It follows,

$$f = \frac{sx_1 + rx_2}{2rs}v + \frac{c(-sx_1 + rx_2)}{2rs}h' + z^* \quad (2.3.11)$$

where  $x_1, x_2, z^*$  satisfy (2.3.8). Considering  $\pmod{\mathbb{Z}v}$ , we finally get

$$N(Y) = \frac{c(-sx_1 + rx_2)}{2rs}h + \frac{sx_1 + rx_2}{d\gamma}u^*(\tilde{H}) + K(H), \quad \text{where } sx_1 + rx_2 \equiv 0 \pmod{d\gamma}. \quad (2.3.12)$$

Let us calculate the lattice  $K(h) = h_{N(Y)}^\perp$ . It is equal to  $(sx_1 + rx_2)/(d\gamma)u^*(\tilde{H}) + K(H)$  where  $sx_1 + rx_2 \equiv 0 \pmod{d\gamma}$  and  $-sx_1 + rx_2 = 0$ . It follows,  $x_1 = ax$ ,  $x_2 = bx$ ,  $x \in \mathbb{Z}$ , and  $sx_1 + rx_2 = 2abcx \equiv 0 \pmod{d\gamma}$ , which is always true. Thus,  $K(h) = [2abc/(d\gamma)u^*(\tilde{H}) + K(H)]$ . By Sect. 2.2,  $u^*(\tilde{H}) + K(H)$  has the order  $2abc^2/(d^2\gamma)$ . We have  $(2abc/(d\gamma), 2abc^2/(d^2\gamma)) = 2abc/(d^2\gamma)(d, c)$  where  $2abc/(d^2\gamma) \in \mathbb{N}$  and

$(d, c) = 1$ . It follows,  $K(h) = [K(H), 2abc/(d^2\gamma)u^*(\tilde{H})]$ , and the index  $[K(h) : K(H)] = c$ .

Let us show that  $d|h$  in  $N(Y)$  and  $h/d$  is primitive in  $N(Y)$ . By (2.3.12), the primitive submodule in  $N(Y)$ , generated by  $h$ , is  $c(-sx_1 + rx_2)/(2rs)h$  where  $(sx_1 + rx_2)/(d\gamma)u^*(\tilde{H}) \in K(h)$  and  $sx_1 + rx_2 \equiv 0 \pmod{d\gamma}$ . From calculation of  $K(h)$  above, we then get  $(sx_1 + rx_2)/(d\gamma) \equiv 0 \pmod{2abc/(d^2\gamma)}$ . It follows,  $bx_1 + ax_2 \equiv 0 \pmod{2ab/d}$ . Let  $d = d_a d_b$  where  $d_a|a$  and  $d_b|b$ . Then  $(a/d_a)|x_1$  and  $(b/d_b)|x_2$ . It follows,  $x_1 = (a/d_a)\tilde{x}_1$  and  $x_2 = (b/d_b)\tilde{x}_2$  where  $\tilde{x}_1, \tilde{x}_2 \in \mathbb{Z}$  and  $d_b\tilde{x}_1 + d_a\tilde{x}_2 \equiv 0 \pmod{2}$ . It follows that the module  $c(-sx_1 + rx_2)/(2rs)h = (-d_b\tilde{x}_1 + d_a\tilde{x}_2)/(2d_a d_b)$  where  $d_b\tilde{x}_1 + d_a\tilde{x}_2 \equiv 0 \pmod{2}$ . It follows that this module is  $\mathbb{Z}(h/d)$ . It proves the statement. We denote

$$\tilde{h} = \frac{h}{d}. \quad (2.3.13)$$

We had proved that  $\tilde{h}$  is primitive in  $N(Y)$  and  $h = d\tilde{h}$ . We have  $\tilde{h}^2 = 2ab/d^2$ .

Let us show that  $\gamma(\tilde{h}) = \gamma(H) = \gamma$ . We remind that  $\gamma(\tilde{h})\mathbb{Z} = \tilde{h} \cdot N(Y)$ . By Sect. 2.2 and (2.3.12),  $\tilde{h}^2/\gamma(\tilde{h})$  is equal to the index  $[(sx_1 + rx_2)/(d\gamma)u^*(\tilde{H}) + K(H)] : K(h)$  where  $sx_1 + rx_2 \equiv 0 \pmod{d\gamma}$ . Such elements  $x_1, x_2$  give  $cd\gamma\mathbb{Z}$ . Thus,  $[(sx_1 + rx_2)/(d\gamma)u^*(\tilde{H}) + K(H)] = [cu^*(\tilde{H}) + K(H)]$ . We had proved that  $K(h) = [2abc/(d^2\gamma)u^*(\tilde{H}) + K(H)]$ . Thus, the index is equal to  $2ab/d^2\gamma$ . Then we get  $\tilde{h}^2/\gamma(\tilde{h}) = 2ab/(d^2\gamma(\tilde{h})) = 2ab/(d^2\gamma)$ . It follows  $\gamma(\tilde{h}) = \gamma$ . We had also proved that  $u^*(\tilde{h}) = m cu^*(H) + K(h)$  where  $m$  is defined  $\pmod{2ab/(d^2\gamma)}$ . We shall calculate  $m$  below.

Now we can rewrite (2.3.12) in the form (2.2.2). We denote  $\tilde{h}^* = \tilde{h}/\tilde{h}^2 = \tilde{h}/(2ab/d^2)$ . We have

$$N(Y) = \frac{-bx_1 + ax_2}{d}\tilde{h}^* + \frac{(bx_1 + ax_2)}{d\gamma}cu^*(\tilde{H}) + K(H) \quad (2.3.14)$$

where  $bx_1 + ax_2 \equiv 0 \pmod{d\gamma}$ . We have proved that the elements  $(-bx_1 + ax_2)/d$  give  $\mathbb{Z}\gamma$ .

Let us write  $d = d_a d_b$  where  $d_a|a$  and  $d_b|b$ . Since  $\gamma|2ab/d^2$  and  $\text{g.c.d}(a, b) = 1$ , we can write  $\gamma = \gamma_2 \gamma_a \gamma_b$  where  $\gamma_a = \text{g.c.d}(\gamma, a/d_a^2)$ ,  $\gamma_b = \text{g.c.d}(\gamma, b/d_b^2)$  and  $\gamma_2 = \gamma/(\gamma_a \gamma_b)$ . Clearly,  $\gamma_2|2$ .

We define  $m = m(a, b, d, \gamma) \pmod{2ab/(d^2\gamma)}$  by the conditions

$$m \equiv -1 \pmod{2a/(d_a^2 \gamma_a \gamma_2)} \quad \text{and} \quad m \equiv 1 \pmod{2b/(d_b^2 \gamma_b \gamma_2)}. \quad (2.3.15)$$

The congruences (2.3.15) define  $m$  uniquely  $\pmod{2ab/(d^2\gamma)}$ . Really, assume that  $x \equiv 0 \pmod{2b/(d_b^2 \gamma_b \gamma_2)}$  and  $x \equiv 0 \pmod{2a/(d_a^2 \gamma_a \gamma_2)}$ . Then  $x/(2/\gamma_2) \equiv 0 \pmod{b/(d_b^2 \gamma_b)}$  and  $x/(2/\gamma_2) \equiv 0 \pmod{a/(d_a^2 \gamma_a)}$ . It follows that  $x/(2/\gamma_2) \equiv 0 \pmod{ab/(d^2 \gamma_a \gamma_b)}$ . Thus,  $x \equiv 0 \pmod{2ab/(d^2 \gamma_a \gamma_b \gamma_2)}$  where  $2ab/(d^2 \gamma_a \gamma_b \gamma_2) =$

$2ab/(d^2\gamma)$ . Thus  $x \equiv 0 \pmod{2ab/(d^2\gamma)}$ . Clearly, there exists a unique  $m(a, b) \pmod{2ab}$  defined by the condition

$$m(a, b) \equiv -1 \pmod{2a} \text{ and } m(a, b) \equiv 1 \pmod{2b}. \quad (2.3.16)$$

Then

$$m(a, b, d, \gamma) \equiv m(a, b) \pmod{\frac{2ab}{d^2\gamma}}, \quad (2.3.17)$$

and one can take (2.3.16) and (2.3.17) as definition of  $m(a, b, d, \gamma)$ .

Let us prove that  $u^*(\tilde{h}) + K(h) = m(a, b, d, \gamma)cu^*(\tilde{H}) + K(h)$ . We had proved that  $u^*(\tilde{h}) + K(h) = m cu^*(\tilde{H}) + K(h)$  where  $m$  is defined  $\pmod{2ab/(d^2\gamma)}$ . To find  $m \pmod{2ab/(d^2\gamma)}$ , one should put  $(-bx_1 + ax_2)/d = \gamma$  in (2.3.14) or

$$-bx_1 + ax_2 = d\gamma. \quad (2.3.18)$$

From (2.3.18) we have  $d_a\gamma_a|x_1$  and  $d_b\gamma_b|x_2$ . From (2.3.14) and (2.3.18), we get

$$m \equiv \frac{ax_2 + bx_1}{d\gamma} \equiv 1 + \frac{2bx_1}{d\gamma} \pmod{2ab/(d^2\gamma)}, \quad (2.3.19)$$

and

$$m \equiv -1 + \frac{2ax_2}{d\gamma} \pmod{2ab/(d^2\gamma)}. \quad (2.3.20)$$

Since  $d_a\gamma_a|x_1$ , we get from (2.3.19) that  $m \equiv 1 \pmod{2b/(d_b^2\gamma_b\gamma_2)}$ , and from (2.3.20) that  $m \equiv -1 \pmod{2a/(d_a^2\gamma_a\gamma_2)}$ . Thus,  $m = m(a, b, d, \gamma) \pmod{2ab/(d^2\gamma)}$ . It proves the statement.

Since  $h^2 = 2ab$  and  $H^2 = 2abc^2$ , we can formally put  $h = H/c$ . By our construction,  $K(H) \subset K(h)$  is a sublattice. Thus, we can consider  $N(X)$  and  $N(Y)$  as extensions of finite index of a common sublattice  $\mathbb{Z}\tilde{H} + K(H)$ .

Finally, we get the very important for us

**Proposition 2.3.1.** *The Picard lattice of  $X$  is*

$$N(X) = [\tilde{H}, K(H), \gamma\tilde{H}^* + u^*(\tilde{H})] \quad (2.3.21)$$

where  $\tilde{H} = H/d$  is primitive in  $N(X)$  with  $H^2 = 2abc^2$  and  $\tilde{H}^* = d^2\tilde{H}/(2abc^2)$ , the lattice  $K(H) = H_{N(X)}^\perp$ , and  $u^*(\tilde{H}) + K(H)$  has the order  $2abc^2/(d^2\gamma)$  in  $K(H)^*/K(H)$ .

The Picard lattice of  $Y$  is

$$N(Y) = [\tilde{h} = h/d, K(h), \gamma\tilde{h}^* + u^*(\tilde{h})], \quad (2.3.22)$$

where the element  $h = (-a, 0, b) \pmod{\mathbb{Z}v}$ ,  $h^2 = 2ab$  and  $\tilde{h} = h/d$  is primitive in  $N(Y)$ , the element  $\tilde{h}^* = d^2\tilde{h}/(2ab)$ , and  $u^*(\tilde{h}) + K(h)$  has the order  $2ab/(d^2\gamma)$  in  $K(h)^*/K(h)$ .

They are related as follows:

$$K(h) = h_{N(Y)}^\perp = [K(H), \frac{2ab}{d^2\gamma} cu^*(\tilde{H})], \quad (2.3.23)$$

and

$$u^*(\tilde{h}) + K(h) = m(a, b)cu^*(\tilde{H}) + K(h) \quad (2.3.24)$$

where  $m(a, b) \bmod 2ab$  is defined by  $m(a, b) \equiv -1 \bmod 2a$  and  $m(a, b) \equiv 1 \bmod 2b$ . To define  $u^*(\tilde{h}) + K(h)$  above, it is enough to consider  $m(a, b) \bmod 2ab/(d^2\gamma)$ .

We can formally put  $h = H/c$ , equivalently  $\tilde{h} = \tilde{H}/c$ . Then  $N(X)$  and  $N(Y)$  become the extensions of a common sublattice:

$$N(X) \supset [\tilde{H} = c\tilde{h}, K(H)] \subset N(Y). \quad (2.3.25)$$

Since  $n(v) = 1$ , the transcendental lattices  $T(X) = T(Y)$  are canonically identified in  $v^\perp$ . It follows that we have the canonical identifications

$$N(X)^*/N(X) = T(X)^*/T(X) = T(Y)^*/T(Y) = N(Y)^*/N(Y). \quad (2.3.26)$$

Here we use that discriminant groups of orthogonal complements in a unimodular lattice are canonically isomorphic. For example, here the identification  $N(X)^*/N(X) = T(X)^*/T(X)$  is given by  $n^* + N(X) \rightarrow t^* + T(X)$ , if  $n^* + t^* \in H^2(X, \mathbb{Z})$ .

Let us calculate the identification

$$N(X)^*/N(X) = N(Y)^*/N(Y). \quad (2.3.27)$$

Obviously (from the description above), it is given by the canonical maps

$$N(X)^* \leftarrow (U \oplus N(X))^* \supset (U \oplus N(X))^*_{v^\perp} \rightarrow (v^\perp)_0^* \leftarrow (v^\perp/\mathbb{Z}v)^*. \quad (2.3.28)$$

Here  $(U \oplus N(X))^*_{v^\perp} = \{x \in (U \oplus N(X))^* | x \cdot v = 0\}$ ,  $(v^\perp) = \{x \in U \oplus N(X) | x \cdot v = 0\}$  and  $(v^\perp)_0^* = \{x \in (v^\perp)^* | x \cdot v = 0\}$ .

Let  $f^* = n\tilde{H}^* + k^* \in N(X)^*$  where  $n \in \mathbb{Z}$  and  $k^* \in K(H)^*$ . The element  $\tilde{f}^* = x_1e_1 + x_2e_2 + n\tilde{H}^* + k^*$  is its lift to  $(U \oplus N(X))^*$  where  $x_1, x_2 \in \mathbb{Z}$ . We have  $\tilde{f}^* \in (U \oplus N(X))^*_{v^\perp}$  if  $-sx_1 - rx_2 + dn = 0$ . It follows that  $n = c(bx_1 + ax_2)/d$  where  $bx_1 + ax_2 \equiv 0 \bmod d$ , and  $\tilde{f}^* = x_1e_1 + x_2e_2 + (c(bx_1 + ax_2)/d)\tilde{H}^* + k^*$ . It follows that

$$f^* = \frac{c(bx_1 + ax_2)}{d}\tilde{H}^* + k^*, \quad bx_1 + ax_2 \equiv 0 \bmod d. \quad (2.3.29)$$

Like in (2.3.11), (2.3.12) and (2.3.14), we finally get that the corresponding to  $f^* + N(X)$  element in  $N(Y)^*/N(Y)$  is  $\epsilon(f^*) + N(Y)$  where

$$\epsilon(f^*) = \frac{-bx_1 + ax_2}{d} \tilde{h}^* + k^* \in N(Y)^*. \quad (2.3.30)$$

Thus, the identification (2.3.27) is given by  $f^* = c((bx_1 + ax_2)/d) \tilde{H}^* + k^* + N(X) \rightarrow \epsilon(f^*) = (-bx_1 + ax_2) \tilde{h}^* + k^* + N(Y)$  where  $bx_1 + ax_1 \equiv 0 \pmod{d}$ . We have  $x_1 = d_a \tilde{x}_1$  and  $x_2 = d_b \tilde{x}_2$  where  $\tilde{x}_1, \tilde{x}_2 \in \mathbb{Z}$ . Then

$$f^* = c((b/d_b) \tilde{x}_1 + (a/d_a) \tilde{x}_2) \tilde{H}^* + k^* \rightarrow \epsilon(f^*) = (-(b/d_b) \tilde{x}_1 + (a/d_a) \tilde{x}_2) \tilde{h}^* + k^*.$$

Let us denote  $n_1 = (b/d_b) \tilde{x}_1 + (a/d_a) \tilde{x}_2$  and  $n_2 = -(b/d_b) \tilde{x}_1 + (a/d_a) \tilde{x}_2$ . We have  $n_1 \equiv m(a, b) n_2 \pmod{2ab/d^2}$ . Really,  $n_1 - m(a, b) n_2 \equiv (2a/d_a) \tilde{x}_2 \equiv 0 \pmod{2a/d_a^2}$  because  $m(a, b) \equiv -1 \pmod{2a/d_a^2}$ . We have  $n_1 - m(a, b) n_2 \equiv (2b/d_b) \tilde{x}_1 \equiv 0 \pmod{2b/d_b^2}$  because  $m(a, b) \equiv 1 \pmod{2b/d_b^2}$ . It follows the statement. Here we consider  $m(a, b) \pmod{2ab/d^2}$ .

The expression  $(b/d_b) \tilde{x}_1 + (a/d_a) \tilde{x}_2$  gives all integers  $n \in \mathbb{Z}$ . Thus, we have proved that if  $f^* = cn \tilde{H}^* + k^* \in N(X)^*$  where  $n \in \mathbb{Z}$ ,  $k^* \in K(H)^*$ , then  $\epsilon(f^*) = m(a, b) \tilde{h}^* + k^* \in N(Y)$ . Thus, we have proved

**Proposition 2.3.2.** *In notations of Proposition 2.3.1, the canonical identification  $\epsilon : N(X)^*/N(X) \cong N(Y)^*/N(Y)$  of the discriminant groups given by periods:*

$$\epsilon : N(X)^*/N(X) = T(X)^*/T(X) = T(Y)^*/T(Y) = N(Y)^*/N(Y) \quad (2.3.31)$$

is given by

$$\epsilon : cn \tilde{H}^* + k^* + N(X) \mapsto m(a, b) n \tilde{h}^* + k^* \quad (2.3.32)$$

where  $n \in \mathbb{Z}$ ,  $k^* \in K(H)^*$ . Here one should consider  $m(a, b) \pmod{2ab/d^2}$ .

Equivalently, the characteristic maps  $\kappa(\tilde{H}) : \tilde{K}(H) \rightarrow N(X)^*/N(X)$  and  $\kappa(\tilde{h}) : \tilde{K}(h) \rightarrow N(Y)^*/N(Y)$  are related as follows:

$$\epsilon(\kappa(\tilde{H})((cn, k^*) + \tilde{K}(H)_0)) = \kappa(\tilde{h})(m(a, b)n, k^*) + \tilde{K}(h)_0 \quad (2.3.33)$$

(here one should consider  $m(a, b) \pmod{2ab/d^2}$ ). Here we set  $\tilde{K}(H) = \tilde{K}(\tilde{H})$  and  $\tilde{K}(h) = \tilde{K}(\tilde{h})$  (see (2.2.7)).

This finishes the calculation of the periods of  $N(Y)$  in terms of the periods of  $N(X)$ .

Applying Lemma 2.2.1 and Propositions 2.3.1 and 2.3.2, by Global Torelli Theorem for K3 surfaces [PS], we get



**Theorem 2.3.3.** *Assume that  $X$  is a K3 surface with a polarization  $H$  with  $H^2 = 2rs$ ,  $r, s \geq 1$ , and a primitive Mukai vector  $v = (r, H, s)$  with the invariants*

$$(a, b, c, d, \gamma) \quad (2.3.34)$$

*introduced above, i.e.  $\tilde{H} = H/d$  is primitive. Let  $Y$  be the moduli space of coherent sheaves on  $X$  with the Mukai vector  $v = (r, H, s)$ . We denote by  $(K(\tilde{H}), u^*(\tilde{H}), \kappa(\tilde{H}))$  the invariants (2.2.11) of  $\tilde{H} \in N(X)$ .*

*The transcendental periods  $(T(X), H^{2,0}(X))$  and  $(T(Y), H^{2,0}(Y))$  are isomorphic if and only if*

$$n(v) = g.c.d(c, d\gamma) = 1 \quad (2.3.35)$$

*(this is Mukai's result). We denote by  $(K(H) = K(\tilde{H}), u^*(\tilde{H}), \kappa(\tilde{H}))$  the invariants (2.2.11) of  $\tilde{H} \in N(X)$ .*

*Assume that (2.3.35) is valid. Then  $Y \cong X$ , if the following conditions (a), (b) and (c) are valid:*

*(a) there exists a primitive  $\tilde{h} \in N(X)$  with  $\tilde{h}^2 = 2ab/d^2$  and  $\gamma(\tilde{h}) = \gamma$ .*

*(b) There exists an embedding  $\phi : K(H) \subset K(\tilde{h})$  of lattices such that  $\phi^*(K(\tilde{h})) = [K(H), 2abc/(d^2\gamma)u^*(\tilde{H})]$  and  $\phi^*(u^*(\tilde{h})) + \phi^*(K(\tilde{h})) = m(a, b)cu^*(\tilde{H}) + \phi^*(K(\tilde{h}))$ . Here  $m(a, b) \bmod 2ab/(d^2\gamma)$  is considered.*

*(c) There exists a choice of  $\pm$  such that  $\kappa(\tilde{h})(m(a, b)n, z^*) = \pm\kappa(H)(cn, \phi^*(z^*))$  if  $(cn, \phi^*(z^*)) \in \tilde{K}(H)^* = \tilde{K}(\tilde{H})^*$ . Here  $m(a, b) \bmod 2ab/d^2$  is considered.*

*The conditions (a), (b) and (c) are necessary for a K3 surface  $X$  with  $\rho(X) \leq 19$  which is general for its Picard lattice  $N(X)$  in the following sense: the automorphism group of the transcendental periods  $(T(X), H^{2,0}(X))$  is  $\pm 1$ . If  $\rho(X) = 20$ , then always  $Y \cong X$  if (2.3.35) holds.*

**Example 2.3.4.** Let us assume that  $\rho(X) = 1$ . Thus,  $N(X) = \mathbb{Z}\tilde{H}$ . Assume conditions (a), (b) and (c) of Theorem 2.3.3 satisfy. Then  $\tilde{h} = \pm\tilde{H}$ . It follows  $c = 1$ . The lattices  $K(H)$  and  $K(\tilde{h})$  are zero, and then (b) is valid. The discriminant group  $N(X) = \mathbb{Z}\tilde{H}^*/\mathbb{Z}\tilde{H} \cong \mathbb{Z}/(2ab/d^2)\mathbb{Z}$ . The condition (c) is valid if and only if  $m(a, b) \equiv \pm 1 \bmod 2ab/d^2$ . This is true if and only if  $(a/d_a^2) = 1$  or  $(b/d_b^2) = 1$ .

Thus,  $X \cong Y$  if  $c = 1$  and either  $a_1 = a/d_a^2 = 1$  or  $b_1 = b/d_b^2 = 1$ . These conditions are necessary to have  $Y \cong X$ , if  $X$  is a general K3 surface with  $\rho(X) = 1$ . We recover the result of Mukai from [Mu3].

**Example 2.3.5.** Following Example 2.3.5, let us consider the case when we can satisfy conditions of Theorem 2.3.3 taking  $\tilde{h} = \pm H$  and  $\phi = \pm id$ . Again we get  $c = 1$ . (b) satisfies, if and only if  $m(a, b) \equiv \pm 1 \bmod 2ab/(d^2\gamma)$ . This is equivalent to either  $2a/(d_a^2\gamma_a\gamma_2) \leq 2$  or  $2b/(d_b^2\gamma_b\gamma_2) \leq 2$ . (c) satisfies, if and only if  $m(a, b) \equiv \pm 1 \bmod 2ab/d^2$ . Thus either  $a/d_a^2 = 1$  or  $b/d_b^2 = 1$ . Thus, always  $Y \cong X$ , if  $c = 1$  and either  $a_1 = a/d_a^2 = 1$  or  $b_1 = b/d_b^2 = 1$ . This is just a specialization (see Lemma 2.1.1) of the  $\rho = 1$  case above. This result is also due to Mukai [Mu3]

In Sect. 3 we consider  $\rho = 2$ . We shall analyse when we can satisfy conditions of Theorem 2.3.3 in this case. By specialization (see Lemma 2.1.1) of these cases, we shall get results about K3 surfaces with any Picard number  $\rho \geq 2$ .

### 3. CONDITIONS OF $Y \cong X$ FOR A GENERAL K3 SURFACE $X$ WITH $\rho = 2$

**3.1. Main results for  $\rho(X) = 2$ .** Here we apply results of Sect. 2 to  $X$  and  $Y$  with Picard number 2. Thus, we assume that  $\rho(X) = \text{rk } N(X) = 2$ .

We start with some preliminary considerations on a primitive element  $P \in S$  of an even hyperbolic lattice  $S$  of  $\text{rk } S = 2$ . We assume that  $P^2 = 2n$ ,  $n \in \mathbb{N}$ , and  $\gamma(P) = \gamma|2n$ .

Let

$$K(P) = P_S^\perp = \mathbb{Z}f(P) \quad (3.1.1)$$

and  $f(P)^2 = -t$  where  $t > 0$  is even. Then  $\pm f(P) \in S$  is defined uniquely by  $P$ . Below we set  $f = f(P)$ .

By elementary considerations, we have

$$S = [\mathbb{Z}P, \mathbb{Z}f, \frac{\gamma(\mu P + f)}{2n}] \quad (3.1.2)$$

where

$$\text{g.c.d}(\mu, \frac{2n}{\gamma}) = 1. \quad (3.1.3)$$

The element

$$\pm \mu \pmod{\frac{2n}{\gamma}} \in (\mathbb{Z}/\frac{2n}{\gamma})^* \quad (3.1.4)$$

is the invariant of the pair  $P \in S$  up to isomorphisms of lattices with a primitive vector  $P$  of  $P^2 = 2n$  and  $\gamma(P) = \gamma$ . If  $f$  changes to  $-f$ , then  $\mu \pmod{2n/\gamma}$  changes to  $-\mu \pmod{2n/\gamma}$ .

We have  $(\gamma(\mu P + f)/(2n))^2 = \gamma^2(\mu^2 - t/2n)/2n \equiv 0 \pmod{2}$ . It follows  $2n\mu^2 - t \equiv 0 \pmod{8n^2/\gamma^2}$ . It follows that for some  $\delta \in \mathbb{N}$  we have

$$f^2 = -\frac{2n\delta}{\gamma}, \quad n\delta \equiv 0 \pmod{\gamma}, \quad \text{and} \quad \delta \equiv \mu^2\gamma \pmod{\frac{4n}{\gamma}}. \quad (3.1.5)$$

We have

$$\det S = -\gamma\delta. \quad (3.1.6)$$

Any element  $z \in S$  can be written as  $z = \gamma(xP + yf)/2n$  where  $x \equiv \mu y \pmod{2n/\gamma}$ . We have

$$z^2 = \frac{\gamma x^2 - \delta y^2}{(2n/\gamma)} \quad (3.1.7)$$

It is convenient to put

$$\tilde{n} = \frac{2n}{\gamma}. \quad (3.1.8)$$

Thus, the considered above case of a primitive  $P \in S$  where  $S$  is an even hyperbolic lattice of  $\text{rk } S = 2$  is described by the invariants

$$\tilde{n}, \gamma, \delta, \pm\mu \in (\mathbb{Z}/\tilde{n})^*, \quad (3.1.9)$$

where  $\tilde{n}, \gamma, \delta \in \mathbb{N}$ . The invariants (3.1.9) must satisfy

$$\tilde{n}\gamma \equiv \tilde{n}\delta \equiv 0 \pmod{2}, \quad \delta \equiv \mu^2\gamma \pmod{2\tilde{n}}. \quad (3.1.10)$$

Then  $P^2 = \tilde{n}\gamma$ ,  $f^2 = -\tilde{n}\delta$ ,  $P \perp f$ , and

$$S = \left\{ \frac{xP + yf}{\tilde{n}} \mid x, y \in \mathbb{Z}, x \equiv \mu y \pmod{\tilde{n}} \right\}. \quad (3.1.11)$$

We have

$$z^2 = \frac{\gamma x^2 - \delta y^2}{\tilde{n}}. \quad (3.1.12)$$

Moreover,

$$\det S = -\delta\gamma. \quad (3.1.13)$$

We denote

$$P^* = \frac{P}{\tilde{n}\gamma}, \quad f^* = \frac{f}{\tilde{n}\delta}. \quad (3.1.14)$$

Then

$$S^* = \{vP^* + wf^* \mid \mu v - w \equiv 0 \pmod{\tilde{n}}\} \quad (3.1.15)$$

and

$$S = \{vP^* + wf^* \mid v \equiv 0 \pmod{\gamma}, w \equiv 0 \pmod{\delta}, \frac{v}{\gamma} \equiv \frac{\mu w}{\delta} \pmod{\tilde{n}}\}. \quad (3.1.16)$$

Here  $v, w \in \mathbb{Z}$ . From (3.1.15),  $w = \mu v + \tilde{n}t$ , and

$$S^* = v(P^* + \mu f^*) + t\tilde{n}f^*, \quad v, t \in \mathbb{Z}, \quad (3.1.17)$$

and  $v(P^* + \mu f^*) + t\tilde{n}f^* \in S$  if and only if

$$v \equiv 0 \pmod{\gamma}, \quad \mu v + \tilde{n}t \equiv 0 \pmod{\delta}, \quad \delta v \equiv \gamma\mu(\mu v + \tilde{n}t) \pmod{\delta\gamma\tilde{n}}. \quad (3.1.18)$$

We have

$$u^*(P) = \frac{\mu^{-1}f}{\tilde{n}} + \mathbb{Z}f = \mu^{-1}\delta f^* + \mathbb{Z}f. \quad (3.1.19)$$

We remind notations we have used in Sect. 2:

$$a_1 = \frac{a}{d_a^2}, \quad b_1 = \frac{b}{d_b^2} \quad (3.1.20)$$

where  $d_a = \text{g.c.d}(d, a)$  and  $d_b = \text{g.c.d}(d, b)$ . We put

$$a_2 = \frac{a_1}{\gamma_a}, \quad b_2 = \frac{b_1}{\gamma_b}, \quad e_2 = \frac{2}{\gamma_2}. \quad (3.1.21)$$

where  $\gamma_a = \text{g.c.d}(a_1, \gamma)$ ,  $\gamma_b = \text{g.c.d}(b_1, \gamma)$ ,  $\gamma_2 = \gamma/(\gamma_a\gamma_b)$ . Then  $\gamma_2|2$ .

Applying calculations above to  $S = N(X)$  and primitive  $P = \tilde{H}$  with  $n = 2a_1b_1c^2$ ,  $\tilde{n} = 2a_1b_1c^2/\gamma = e_2a_2b_2c^2$ , and  $\gamma(\tilde{H}) = \gamma$ , we get

**Proposition 3.1.1.** *Let  $X$  be a K3 surface with Picard number  $\rho = 2$  equipped with a primitive polarization (or vector)  $\tilde{H} \in N(X)$  of degree  $\tilde{H}^2 = 2a_1b_1c^2$  and  $\gamma(\tilde{H}) = \gamma|2a_1b_1$ .*

*Let  $K(\tilde{H}) = (\tilde{H})^\perp = \mathbb{Z}f(\tilde{H})$ . We have  $f(\tilde{H})^2 = -2a_1b_1c^2\delta/\gamma$  where  $\det N(X) = -\delta\gamma$ .*

*For some  $\mu \in (\mathbb{Z}/(2a_1b_1c^2/\gamma))^*$  (the  $\pm\mu$  is the invariant of the pair  $\tilde{H} \in N(X)$ ) where  $\delta \equiv \mu^2\gamma \pmod{4a_1b_1c^2/\gamma}$ , one has*

$$N(X) = [\tilde{H}, f(\tilde{H}), \frac{(\mu\tilde{H} + f(\tilde{H}))}{2a_1b_1c^2/\gamma}], \quad (3.1.22)$$

$$N(X) = \{z = \frac{x\tilde{H} + yf(\tilde{H})}{2a_1b_1c^2/\gamma} \mid x, y \in \mathbb{Z} \text{ and } x \equiv \mu y \pmod{\frac{2a_1b_1c^2}{\gamma}}\}. \quad (3.1.23)$$

We have

$$z^2 = \frac{\gamma x^2 - \delta y^2}{2a_1b_1c^2/\gamma}. \quad (3.1.24)$$

For any primitive element  $P \in N(X)$  with  $P^2 = 2a_1b_1c^2$ ,  $\gamma(P) = \gamma$  and the same invariant  $\pm\mu$ , there exists an automorphism  $\phi \in O(N(X))$  such that  $\phi(\tilde{H}) = P$ .

Applying calculations above to  $S = N(Y)$  and primitive  $P = \tilde{h} \in N(Y)$  with  $n = 2a_1b_1$ ,  $\tilde{n} = 2a_1b_1/\gamma = e_2a_2b_2$  and  $\gamma(\tilde{h}) = \gamma$ , we get

**Proposition 3.1.2.** *Let  $Y$  be a K3 surface with Picard number  $\rho = 2$  equipped with a primitive polarization (or vector)  $\tilde{h} \in N(Y)$  of degree  $\tilde{h}^2 = 2a_1b_1$  and  $\gamma(\tilde{h}) = \gamma|2a_1b_1$ .*

*Let  $K(\tilde{h}) = (\tilde{h})^\perp = \mathbb{Z}f(\tilde{h})$ . We have  $f(\tilde{h})^2 = -2a_1b_1\delta/\gamma$  where  $\det N(Y) = -\delta\gamma$ .*

*For some  $\nu \in (\mathbb{Z}/(2a_1b_1/\gamma))^*$  (the  $\pm\nu$  is the invariant of the pair  $\tilde{h} \in N(Y)$ ) where  $\delta \equiv \nu^2\gamma \pmod{4a_1b_1/\gamma}$ , one has*

$$N(Y) = [\tilde{h}, f(\tilde{h}), \frac{(\nu\tilde{h} + f(\tilde{h}))}{2a_1b_1/\gamma}], \quad (3.1.25)$$

$$N(Y) = \{z = \frac{x\tilde{h} + yf(\tilde{h})}{2a_1b_1/\gamma} \mid x, y \in \mathbb{Z} \text{ and } x \equiv \nu y \pmod{\frac{2a_1b_1}{\gamma}}\}. \quad (3.1.26)$$

We have

$$z^2 = \frac{\gamma x^2 - \delta y^2}{2a_1b_1/\gamma}. \quad (3.1.27)$$

For any primitive element  $P \in N(Y)$  with  $P^2 = 2a_1b_1$ ,  $\gamma(P) = \gamma$  and the same invariant  $\pm\nu$ , there exists an automorphism  $\phi \in O(N(Y))$  such that  $\phi(\tilde{h}) = P$ .

The crucial statement is

**Theorem 3.1.3.** *Let  $X$  be a K3 surface,  $\rho(X) = 2$  and  $H$  a polarization of  $X$  of degree  $H^2 = 2rs$ ,  $r, s \geq 1$ , and Mukai vector  $(r, H, s)$  is primitive. Let  $Y$  be the moduli space of sheaves on  $X$  with the isotropic Mukai vector  $v = (r, H, s)$  and the canonical nef element  $h = (-a, 0, b) \bmod \mathbb{Z}v$ . We assume that*

$$g.c.d(c, d\gamma) = 1.$$

*With notations of Propositions 3.1.1, all elements*

$$\tilde{h} = \frac{x\tilde{H} + yf(\tilde{H})}{2a_1b_1c^2/\gamma} \in N(X)$$

*with square  $\tilde{h}^2 = 2a_1b_1$  satisfying Theorem 2.2.3 are in one to one correspondence with integral solutions  $(x, y)$  of the equation*

$$\gamma x^2 - \delta y^2 = 4a_1^2b_1^2c^2/\gamma \quad (3.1.28)$$

*which satisfy conditions (i) — (v) below:*

(i)

$$x \equiv \mu y \bmod 2a_1b_1c^2/\gamma, \quad (3.1.29)$$

$$\mu\gamma x \equiv \delta y \bmod 2a_1b_1c^2; \quad (3.1.30)$$

*(ii)  $(x, y)$  belongs to one of a-series (the sign  $+$ ) or b-series (the sign  $-$ ) of solutions defined below:*

$$\pm m(a, b)\mu x + (\delta y/\gamma) \equiv 0 \bmod 2a_1b_1/\gamma, \quad (3.1.31)$$

$$x \pm m(a, b)\mu y \equiv 0 \bmod 2a_1b_1/\gamma, \quad (3.1.32)$$

$$\pm m(a, b)\mu x + (\delta y/\gamma) \equiv \mu(x \pm m(a, b)\mu y) \bmod (2a_1b_1c^2/\gamma)(2a_1b_1/\gamma); \quad (3.1.33)$$

*(iii) there exists a choice of  $\beta = \pm 1$  such that*

$$\left\{ \begin{array}{l} \left( \frac{m(a, b)\gamma x \pm \mu\gamma y}{2a_1b_1} - \beta c \right) \equiv 0 \bmod \gamma \\ \left( \frac{\delta m(a, b)y \pm \mu\gamma x}{2a_1b_1} - \beta\mu c \right) \equiv 0 \bmod \delta \\ \delta \left( \frac{m(a, b)\gamma x \pm \mu\gamma y}{2a_1b_1} - \beta c \right) \equiv \mu\gamma \left( \frac{\delta m(a, b)y \pm \mu\gamma x}{2a_1b_1} - \beta\mu c \right) \bmod 2a_1b_1c^2\delta \end{array} \right. \quad (3.1.34)$$

*and*

$$\left\{ \begin{array}{l} c\delta y \equiv 0 \bmod \gamma \\ cx - (\pm\beta)\frac{2a_1b_1c^2}{\gamma} \equiv 0 \bmod \delta \\ \delta y \equiv \mu(\gamma x - (\pm\beta)2a_1b_1c) \bmod 2a_1b_1c\delta \end{array} \right. \quad (3.1.35)$$

*where  $+$  is taken for a-series, and  $-$  is taken for b-series.*

(iv) the pair  $(x, y)$  is  $\mu$ -primitive:

$$g.c.d\left(x, y, \frac{x - \mu y}{2a_1b_1c^2/\gamma}\right) = 1; \quad (3.1.36)$$

(v)  $\gamma(\tilde{h}) = \gamma$ , equivalently

$$g.c.d\left(\gamma x, \delta y, \frac{\mu\gamma x - \delta y}{2a_1b_1c^2/\gamma}\right) = \gamma. \quad (3.1.37)$$

In particular (by Theorem 2.3.3), for a general  $X$  with  $\rho(X) = 2$  we have  $Y \cong X$  if and only if the equation  $\gamma x^2 - \delta y^2 = 4a_1^2b_1^2c^2/\gamma$  has an integral solution  $(x, y)$  satisfying conditions (i)–(v) above. Moreover, a nef primitive element  $P = (x\tilde{H} + f(\tilde{H}))/ (2a_1b_1c^2/\gamma)$  with  $P^2 = 2a_1b_1$  and  $\gamma(P) = \gamma$  defines a pair  $(X, P)$  which is isomorphic to the  $(Y, \tilde{h})$  if and only if  $(x, y)$  satisfies the conditions (ii) and (iii) (it satisfies conditions (i), (iv) and (v) since it corresponds to a primitive element of  $N(X)$  with  $\gamma(P) = \gamma$ ).

*Proof.* We denote

$$H^* = \frac{\tilde{H}}{2a_1b_1c^2}, \quad f(\tilde{H})^* = \frac{\gamma f(\tilde{H})}{2a_1b_1c^2\delta} \quad (3.1.38)$$

where  $K(H) = \mathbb{Z}f(\tilde{H}) = H^\perp$  in  $N(X)$ . By (3.1.19), we have

$$u^*(\tilde{H}) = \frac{\mu^{-1}\gamma f(\tilde{H})}{2a_1b_1c^2} + \mathbb{Z}f(\tilde{H}) = \mu^{-1}\delta f(\tilde{H})^* + \mathbb{Z}f(\tilde{H}). \quad (3.1.39)$$

Let

$$\tilde{h} = \frac{x\tilde{H} + yf(\tilde{H})}{2a_1b_1c^2/\gamma} \in N(X) \quad (3.1.40)$$

satisfies conditions of Theorem 2.2.1. Then  $x, y \in \mathbb{Z}$  and  $x \equiv \mu y \pmod{2a_1b_1c^2/\gamma}$ . We get (3.1.29) in (i). Moreover,  $\tilde{h}$  is primitive which is equivalent to (iv). We also have  $\gamma(\tilde{h}) = \gamma$ . It is equivalent to

$$g.c.d\left(\tilde{H} \cdot \tilde{h}, f(\tilde{H}) \cdot \tilde{h}, (\mu\tilde{H} + f(\tilde{H}))/ (2a_1b_1c^2/\gamma) \cdot \tilde{h}\right) = \gamma.$$

It follows (3.1.30) in (i), and (v). We have  $\tilde{h}^2 = 2a_1b_1$ . This is equivalent to  $\gamma x^2 - \delta y^2 = 4a_1^2b_1^2c^2/\gamma$ .

Consider  $K(\tilde{h}) = \tilde{h}^\perp$  in  $N(X)$ . Let us denote

$$f(\tilde{h}) = \frac{(\delta y/\gamma)\tilde{H} + xf(\tilde{H})}{2a_1b_1c^2/\gamma}. \quad (3.1.41)$$

The element  $f(\tilde{h}) \in N(X)$  because of (i). We have  $f(\tilde{h}) \perp \tilde{h}$  and  $f(\tilde{h})^2 = -2a_1b_1\delta/\gamma$ . Since  $\gamma(\tilde{h}) = \gamma$ , it follows that  $K(\tilde{h}) = \mathbb{Z}f(\tilde{h})$  and  $f(\tilde{h})$  is primitive. We have

$$N(X) = \left[ \tilde{h}, f(\tilde{h}), \frac{\nu\tilde{h} + f(\tilde{h})}{2a_1b_1/\gamma} \right] \quad (3.1.42)$$

where  $\nu \in (\mathbb{Z}/(2a_1b_1/\gamma))^*$  (according to Proposition 3.1.2). We denote

$$\tilde{h}^* = \frac{\tilde{h}}{2a_1b_1}, \quad f(\tilde{h})^* = \frac{\gamma f(\tilde{h})}{2a_1b_1\delta}. \quad (3.1.43)$$

By (3.1.19), we have

$$u^*(\tilde{h}) = \frac{\nu^{-1}f(\tilde{h})}{2a_1b_1/\gamma} + \mathbb{Z}f(\tilde{h}) = \nu^{-1}\delta f(\tilde{h})^* + \mathbb{Z}f(\tilde{h}). \quad (3.1.44)$$

There exists a unique (up to  $\pm 1$ ) embedding

$$\phi : K(\tilde{H}) = \mathbb{Z}f(\tilde{H}) \rightarrow K(\tilde{h}) = \mathbb{Z}f(\tilde{h}), \quad \phi(f(\tilde{H})) = \pm cf(\tilde{h}). \quad (3.1.45)$$

of one-dimensional lattices. Its dual is defined by  $\phi^*(f(\tilde{h})^*) = \pm cf(\tilde{H})^*$ .

We have

$$\begin{aligned} \phi^*(K(\tilde{h})) &= \mathbb{Z}\phi^*(f(\tilde{h})) = \mathbb{Z}f(\tilde{H})/c = \\ [K(\tilde{H}), (2a_1b_1c/\gamma)u^*(\tilde{H})] &= [K(\tilde{H}), (2abc/(d^2\gamma))u^*(\tilde{H})] \end{aligned}$$

because of (3.1.39). This gives the first part of (b) in Theorem 2.3.3.

We have

$$\begin{aligned} \phi^*(u^*(\tilde{h})) + \phi^*(K(\tilde{h})) &= \nu^{-1}\delta\phi^*(f(\tilde{h})^*) + \mathbb{Z}\phi^*(f(\tilde{h})) = \pm\nu^{-1}\delta cf(\tilde{H})^* + \mathbb{Z}f(\tilde{H})/c = \\ &\pm\nu^{-1}\delta cf(\tilde{H})^* + \mathbb{Z}(2a_1b_1c\delta/\gamma)f(\tilde{H})^*. \end{aligned}$$

On the other hand, by (3.1.39)

$$m(a, b)cu^*(\tilde{H}) + \mathbb{Z}(2a_1b_1c\delta/\gamma)f(\tilde{H})^* = m(a, b)\mu^{-1}\delta cf(\tilde{H})^* + \mathbb{Z}(2a_1b_1c\delta/\gamma)f(\tilde{H})^*.$$

Thus, by (3.1.44), second part of (b) in Theorem 2.3.3 is  $\pm\nu^{-1} \equiv m(a, b)\mu^{-1} \pmod{2a_1b_1/\gamma}$ . Equivalently,

$$\nu \equiv \pm m(a, b)\mu \pmod{\frac{2a_1b_1}{\gamma}}. \quad (3.1.46)$$

Thus, for  $\nu$  given by (3.1.46) one has (this is the definition of  $\nu$ )

$$\frac{\nu\tilde{h} + f(\tilde{h})}{2a_1b_1/\gamma} = \left( \frac{\nu x + (\delta y/\gamma)}{2a_1b_1/\gamma} \tilde{H} + \frac{x + \nu y}{2a_1b_1/\gamma} f(\tilde{H}) \right) / (2a_1b_1c^2/\gamma) \in N(X). \quad (3.1.47)$$

This is equivalent for  $\nu \equiv m(a, b)\mu \pmod{2a_1b_1/\gamma}$  to

$$\begin{cases} \pm m(a, b)\mu x + (\delta y/\gamma) \equiv 0 \pmod{2a_1b_1/\gamma} \\ x \pm m(a, b)\mu y \equiv 0 \pmod{2a_1b_1/\gamma} \\ \pm m(a, b)\mu x + (\delta y/\gamma) \equiv \mu(x \pm m(a, b)\mu y) \pmod{(2a_1b_1c^2/\gamma)(2a_1b_1/\gamma)} \end{cases} . \quad (3.1.48)$$

This gives (ii).

Let us consider the condition (c) of Theorem 2.3.3. For a choice of  $\beta = \pm 1$ , one has

$$m(a, b)n\tilde{h}^* + z^* \equiv \beta(cn\tilde{H}^* + \phi^*(z^*)) \pmod{N(X)}, \quad n \in \mathbb{Z}, \quad z \in K(\tilde{h})^* \quad (3.1.49)$$

if

$$cn\tilde{H}^* + \phi^*(z^*) \in N(X)^*. \quad (3.1.50)$$

Let  $z^* = kf(\tilde{h})^*$ ,  $k \in \mathbb{Z}$ , then  $\phi^*(kf(\tilde{h})^*) = \pm kcf(\tilde{H})^*$  and  $cn\tilde{H}^* + \phi^*(z^*) = cn\tilde{H}^* \pm kcf(\tilde{H})^*$ . By (3.1.15),  $cn\tilde{H}^* \pm kcf(\tilde{H})^* \in N(X)^*$  if and only if  $\mu cn \mp kc \equiv 0 \pmod{2a_1b_1c^2/\gamma}$ . This is equivalent to  $k \equiv \pm \mu n \pmod{2a_1b_1c/\gamma}$ . Like in (3.1.17), we get  $k = \pm \mu n + (2a_1b_1c/\gamma)t$  where  $n, t \in \mathbb{Z}$ . We get

$$cn\tilde{H}^* \pm kcf(\tilde{H})^* = cn(\tilde{H}^* + \mu f(\tilde{H})^*) \pm t(2a_1b_1c^2/\gamma)f(\tilde{H})^*. \quad (3.1.51)$$

Thus, it is enough to check

$$m(a, b)n\tilde{h}^* + kf(\tilde{h})^* \equiv \beta(cn\tilde{H}^* \pm kcf(\tilde{H})^*) \pmod{N(X)}, \quad (3.1.52)$$

where  $(n, k) = (1, \pm \mu)$  or  $(n, k) = (0, 2a_1b_1c/\gamma)$ . Thus, one should check for one of  $\beta = \pm 1$  that

$$m(a, b)\tilde{h}^* \pm \mu f(\tilde{h})^* \equiv \beta(c\tilde{H}^* + c\mu f(\tilde{H})^*) \pmod{N(X)} \quad (3.1.53)$$

and

$$\frac{2a_1b_1cf(\tilde{h})^*}{\gamma} \equiv \pm \beta \frac{2a_1b_1c^2f(\tilde{H})^*}{\gamma} \pmod{N(X)}. \quad (3.1.54)$$

We have

$$\begin{aligned} m(a, b)n\tilde{h}^* + kf(\tilde{h})^* &= m(a, b)n \frac{\tilde{h}}{2a_1b_1} + k \frac{\gamma f(\tilde{h})}{2a_1b_1\delta} = \\ &= \frac{1}{2a_1b_1\delta} \left( \delta m(a, b)n\tilde{h} + \gamma kf(\tilde{h}) \right) = \\ &= \frac{1}{2a_1b_1\delta} \left( \delta m(a, b)n \frac{x\tilde{H} + yf(\tilde{H})}{2a_1b_1c^2/\gamma} + \gamma k \frac{(\delta y/\gamma)\tilde{H} + xf(\tilde{H})}{2a_1b_1c^2/\gamma} \right) = \end{aligned}$$



$$\begin{aligned} \frac{1}{2a_1b_1\delta} \left( \delta m(a,b)n(\gamma x \tilde{H}^* + \delta y f(\tilde{H})^*) + \gamma k(\delta y \tilde{H}^* + \delta x f(\tilde{H})^*) \right) = \\ \frac{m(a,b)n\gamma x + \gamma ky}{2a_1b_1} \tilde{H}^* + \frac{\delta m(a,b)ny + \gamma kx}{2a_1b_1} f(\tilde{H})^*. \end{aligned} \quad (3.1.55)$$

Thus, (3.1.53) is

$$\left( \frac{m(a,b)\gamma x \pm \mu\gamma y}{2a_1b_1} - \beta c \right) \tilde{H}^* + \left( \frac{\delta m(a,b)y \pm \mu\gamma x}{2a_1b_1} - \beta\mu c \right) f(\tilde{H})^* \in N(X) \quad (3.1.56)$$

and (3.1.54) is

$$cy\tilde{H}^* + \left( cx - (\pm\beta)\frac{2a_1b_1c^2}{\gamma} \right) f(\tilde{H})^* \in N(X). \quad (3.1.57)$$

Here one should take  $+$  if  $(x, y)$  belongs to  $a$ -series, and one should take  $-$  if  $(x, y)$  belongs to  $b$ -series.

By (3.1.16), we can reformulate (3.1.56) as (3.1.34) in (iii), and we can reformulate (3.1.57) as (3.1.35) in (iii).

This finishes the proof.

Now we analyse conditions of Theorem 3.1.3. The most important are congruences  $\pmod{\delta}$  since  $\delta$  is not bounded by a constant depending on  $(r, s)$ .

Let us consider the congruence  $cx - (\pm\beta)(2a_1b_1c^2/\gamma) \equiv 0 \pmod{\delta}$  in (3.1.35). We know that  $\delta \equiv \gamma\mu^2 \pmod{(2a_1b_1/\gamma)c^2}$  where  $\mu \pmod{(2a_1b_1/\gamma)c^2}$  is invertible,  $\gamma|2a_1b_1$  and  $\text{g.c.d}(\gamma, c) = 1$ . It follows that  $\text{g.c.d}(c, \delta) = 1$ . Thus, the considered congruence is equivalent to

$$x \equiv \pm \frac{2a_1b_1c}{\gamma} \pmod{\delta} \quad (3.1.58)$$

since  $\beta = \pm 1$ . Later we shall see that all congruences  $\pmod{\delta}$  which follow from conditions of Theorem 3.1.3 are consequences of (3.1.58). Thus, (3.1.58) is the most important condition of Theorem 3.1.3.

Let us consider all integral  $(x, y)$  which satisfy (3.1.28) and (3.1.58), i. e.

$$\gamma x^2 - \delta y^2 = \frac{4a_1^2b_1^2c^2}{\gamma} \quad \text{and} \quad x \equiv \pm \frac{2a_1b_1c}{\gamma} \pmod{\delta}. \quad (3.1.59)$$

We apply the main trick used in [MN1], [MN2] and [N4]. Considering  $\pm(x, y)$ , we can assume that  $x \equiv 2a_1b_1c/\gamma \pmod{\delta}$ , i. e.  $x = 2a_1b_1c/\gamma - k\delta$  where  $k \in \mathbb{Z}$ . We have  $\gamma^2x^2 = 4a_1^2b_1^2c^2 - 4a_1b_1ck\delta\gamma + \gamma^2k^2\delta^2 = \gamma\delta y^2 + 4a_1^2b_1^2c^2$ . It follows

$$\delta = \frac{y^2 + 4a_1b_1ck}{\gamma k^2}. \quad (3.1.60)$$

Consider a prime  $l$  such that  $l \nmid 2a_1b_1c$ . Assume that  $l^{2t+1} \mid k$ , but  $l^{2t+2} \nmid k$ . We have  $k \mid y^2$ . Then  $l^{2t+1} \mid y^2$ . Then  $l^{2t+2} \mid y^2$ . Since  $k^2 \mid y^2 + 4a_1b_1ck$ , it follows that  $l^{2t+2} \mid y^2 + 4a_1b_1ck$ . We then get  $l^{2t+2} \mid k$ . We get a contradiction. It follows that  $k = -\alpha q^2$  where  $q \in \mathbb{Z}$  (we can additionally assume that  $q \geq 0$ ),  $\alpha \mid 2a_1b_1c$  and  $\alpha$  is square-free. Remark that  $\alpha$  can be negative.

From (3.1.60), we get

$$\delta = \frac{y^2 - 4a_1b_1c\alpha q^2}{\gamma\alpha^2q^4}. \quad (3.1.61)$$

It follows  $\alpha q \mid y$  and  $y = \alpha pq$  where  $p \in \mathbb{Z}$ . From (3.1.61),

$$\delta = \frac{p^2 - 4a_1b_1c/\alpha}{\gamma q^2}. \quad (3.1.62)$$

Equivalently,

$$p^2 - \gamma\delta q^2 = \frac{4a_1b_1c}{\alpha}. \quad (3.1.63)$$

If integral  $p, q$  satisfy (3.1.63), we have

$$(x, y) = \pm \left( \frac{2a_1b_1c}{\gamma} + \alpha\delta q^2, \alpha pq \right). \quad (3.1.64)$$

satisfy (3.1.59). We call solutions (3.1.64) of (3.1.59) as associated solutions. Thus, we get the very important for us

**Theorem 3.1.4.** *All integral solutions  $(x, y)$  of*

$$\gamma x^2 - \delta y^2 = \frac{4a_1^2b_1^2c^2}{\gamma} \quad \text{and} \quad x \equiv \pm \frac{2a_1b_1c}{\gamma} \pmod{\delta} \quad (3.1.65)$$

*are associated solutions*

$$(x, y) = \pm \left( \frac{2a_1b_1c}{\gamma} + \alpha\delta q^2, \alpha pq \right) \quad (3.1.66)$$

*to integral solutions  $\alpha, (p, q)$  of*

$$\alpha \mid 2a_1b_1c \text{ where } \alpha \text{ is square-free, and } p^2 - \gamma\delta q^2 = \frac{4a_1b_1c}{\alpha}. \quad (3.1.67)$$

*Any solution  $(x, y)$  of (3.1.65) can be written in the form (3.1.66) where  $\alpha, (p, q)$  is solution of (3.1.67). Any solution of (3.1.67) gives a solution (3.1.66) of (3.1.65).*

*Solutions of (3.1.65) and (3.1.67) are in one-to-one correspondence if we additionally assume that  $q \geq 0$ .*

Now we can write  $(x, y)$  of Theorem 3.1.3 in the form (3.1.66) as associated solutions to (3.1.67). Putting that  $(x, y)$  to relations (i)—(v) of Theorem 3.1.3, we get

some relations on  $\alpha$  and  $(p, q)$ . They are a finite number of congruences mod  $N_i$  where  $N_i$  depend only on  $(r, s)$  (or  $(a, b, c)$ ). All  $N_i$  are bounded by functions depending only on  $(r, s)$ . These congruences have a lot of relations between them and with (3.1.67). All together they give many very strong restrictions on  $\alpha$  and  $(p, q)$ . We analyse them below.

We fix

$$\mu \in (\mathbb{Z}/(2a_1b_1c^2/\gamma))^* \quad (3.1.68)$$

and consider  $\delta \in \mathbb{N}$  such that

$$\delta \equiv \mu^2\gamma \pmod{\frac{4a_1b_1c^2}{\gamma}} \quad (3.1.69)$$

The relation (3.1.29) in (i) of Theorem 3.1.3 is equivalent to

$$2a_1b_1c + \alpha\gamma\delta q^2 \equiv \gamma\mu\alpha pq \pmod{2a_1b_1c^2}. \quad (3.1.70)$$

By (3.1.67), we have

$$-4a_1b_1c + \alpha p^2 - \alpha\gamma\delta q^2 = 0. \quad (3.1.71)$$

Taking sum, we get

$$2a_1b_1c \equiv \alpha p(p - \mu\gamma q) \pmod{2a_1b_1c^2}. \quad (3.1.72)$$

The relation (3.1.72) is equivalent to (3.1.29). Taking 2(3.1.70)+(3.1.71), we get

$$\alpha p^2 + \alpha\gamma\delta q^2 \equiv 2\gamma\mu\alpha pq \pmod{4a_1b_1c^2} \quad (3.1.73)$$

which is also equivalent to (3.1.29). From (3.1.69), we get

$$\alpha p^2 + \alpha\mu^2\gamma^2 q^2 \equiv 2\gamma\mu\alpha pq \pmod{4a_1b_1c^2} \quad (3.1.74)$$

and

$$\alpha(p - \mu\gamma q)^2 \equiv 0 \pmod{4a_1b_1c^2}. \quad (3.1.75)$$

This is equivalent to (3.1.73). It follows  $\alpha(p - \mu\gamma q)^2 \equiv 0 \pmod{4c^2}$ . Since  $\alpha$  is square-free, it follows

$$2c \mid (p - \mu\gamma q). \quad (3.1.76)$$

From (3.1.72), we get

$$2a_1b_1 \equiv \alpha p \frac{p - \mu\gamma q}{c} \pmod{2a_1b_1c} \quad (3.1.77)$$

where  $(p - \mu\gamma q)/c$  is an integer. It follows,  $\alpha \mid 2a_1b_1$ . Thus, we get

**Proposition 3.1.5.** *The condition (3.1.29) of Theorem 3.1.3 is equivalent to*

$$\alpha(p - \mu\gamma q)^2 \equiv 0 \pmod{4a_1b_1c^2}. \quad (3.1.78)$$

We have

$$\alpha|2a_1b_1. \quad (3.1.79)$$

The condition (3.1.78) is also equivalent to

$$2a_1b_1c \equiv \alpha p(p - \mu\gamma q) \pmod{2a_1b_1c^2}. \quad (3.1.80)$$

Considering the condition (3.1.30) of Theorem 3.1.3, we similarly get

**Proposition 3.1.6.** *The condition (3.1.30) of Theorem 3.1.3 is equivalent to*

$$\mu\alpha p^2 + \mu\alpha\gamma\delta q^2 \equiv 2\alpha\delta pq \pmod{4a_1b_1c^2}. \quad (3.1.81)$$

Congruences (3.1.78) and (3.1.81) are equivalent  $\pmod{(4a_1b_1c^2/\gamma)}$ , and (3.1.81) is equivalent to (3.1.78) together with

$$2\alpha pq(\delta - \gamma\mu^2) \equiv 0 \pmod{4a_1b_1c^2} \quad (3.1.82)$$

where  $\delta - \gamma\mu^2 \equiv 0 \pmod{4a_1b_1c^2/\gamma}$ .

*Proof.* To get (3.1.82), consider  $\mu$  (3.1.78) - (3.1.81).

Now consider (3.1.31) of Theorem 3.1.3. Let us consider the  $a$ -series (the sign +). Since  $m(a, b) \equiv -1 \pmod{2a_1/(\gamma_2\gamma_a)}$ , we get  $-\mu x + (\delta y/\gamma) \equiv 0 \pmod{2a_1/(\gamma_2\gamma_a)}$ . It is satisfied because of (3.1.30). Since  $m(a, b) \equiv 1 \pmod{2b_1/(\gamma_2\gamma_b)}$ , we get  $\mu x + (\delta y/\gamma) \equiv 0 \pmod{2b_1/(\gamma_2\gamma_b)}$ . By (3.1.30), we get  $\mu x - (\delta y/\gamma) \equiv 0 \pmod{2b_1/(\gamma_2\gamma_b)}$ . Thus, we get  $2\mu x \equiv 0 \pmod{2b_1/(\gamma_2\gamma_b)}$ . If  $\gamma_2 = 1$ , this is equivalent to  $\mu x \equiv 0 \pmod{b_1/\gamma_b}$  and  $x \equiv 0 \pmod{b_1/\gamma_b}$ . If  $\gamma_2 = 2$ , then  $b_1/\gamma_b$  is odd, and we get  $2\mu x \equiv 0 \pmod{(b_1/\gamma_b)}$  which is equivalent to  $x \equiv 0 \pmod{b_1/\gamma_b}$ . Thus, at any case we get  $(b_1/\gamma_b)|x$ , equivalently,  $x \equiv 0 \pmod{(b_1/\gamma_b)}$ . By (3.1.66), this is equivalent to  $\alpha\delta q^2 \equiv 0 \pmod{(b_1/\gamma_b)}$ . We have  $\delta \equiv \mu^2\gamma \pmod{(b_1/\gamma_b)}$  and  $\mu \pmod{(b_1/\gamma_b)}$  is invertible. Thus, we get  $\alpha\gamma q^2 \equiv 0 \pmod{(b_1/\gamma_b)}$  which is equivalent to  $\alpha(\gamma_b q)^2 \equiv 0 \pmod{b_1}$ .

Let us consider (3.1.32) of Theorem 3.1.3. We consider the  $a$ -series. We get  $x - \mu y \equiv 0 \pmod{2a_1/(\gamma_2\gamma_a)}$ . It satisfies because of (3.1.29). We get  $x + \mu y \equiv 0 \pmod{2b_1/(\gamma_2\gamma_b)}$ . Using (3.1.29), we similarly get that this is equivalent to  $(b_1/\gamma_b)|x$ .

Thus, we finally get

**Proposition 3.1.7.** *The conditions (3.1.31) and (3.1.32) in (ii) of Theorem 3.1.3 are equivalent to*

$$x \equiv 0 \pmod{\frac{b_1}{\gamma_b}} \quad (3.1.83)$$

or to

$$\alpha(\gamma_b q)^2 \equiv 0 \pmod{b_1} \quad (3.1.84)$$

for  $a$ -series (the sign  $+$ ), and it is equivalent to

$$x \equiv 0 \pmod{\frac{a_1}{\gamma_a}} \quad (3.1.85)$$

or to

$$\alpha(\gamma_a q)^2 \equiv 0 \pmod{a_1} \quad (3.1.86)$$

for  $b$ -series (the sign  $-$ ).

Consider (3.1.33) in Theorem 3.1.3. We consider the  $a$ -series. We get mod  $2a_1/(\gamma_2\gamma_a)$  that

$$2\mu\gamma x \equiv (2\delta + (\mu^2\gamma - \delta))y \pmod{(2a_1b_1c^2)(2a_1/(\gamma_2\gamma_a))}.$$

We get mod  $2b_1/(\gamma_2\gamma_b)$  that

$$(\delta - \mu^2\gamma)y \equiv 0 \pmod{(2a_1b_1c^2)(2b_1/(\gamma_2\gamma_b))}.$$

Using (3.1.66) (and (3.1.71) to make the relations homogeneous), we finally get

**Proposition 3.1.8.** *The condition (3.1.33) of Theorem 3.1.3 is equivalent to*

$$2\mu\gamma x \equiv (\delta + \mu^2\gamma)y \pmod{(2a_1b_1c^2)(2a_1/(\gamma_2\gamma_a))} \quad (3.1.87)$$

and

$$(\delta - \mu^2\gamma)y \equiv 0 \pmod{(2a_1b_1c^2)(2b_1/(\gamma_2\gamma_b))} \quad (3.1.88)$$

or

$$\mu\alpha p^2 - \alpha(\delta + \mu^2\gamma)pq + \mu\alpha\gamma\delta q^2 \equiv 0 \pmod{(2a_1b_1c^2)(2a_1/(\gamma_2\gamma_a))} \quad (3.1.89)$$

and

$$\alpha(\delta - \mu^2\gamma)pq \equiv 0 \pmod{(2a_1b_1c^2)(2b_1/(\gamma_2\gamma_b))} \quad (3.1.90)$$

for the  $a$ -series (the sign  $+$ ), and it is equivalent to

$$2\mu\gamma x \equiv (\delta + \mu^2\gamma)y \pmod{(2a_1b_1c^2)(2b_1/(\gamma_2\gamma_b))} \quad (3.1.91)$$

and

$$(\delta - \mu^2\gamma)y \equiv 0 \pmod{(2a_1b_1c^2)(2a_1/(\gamma_2\gamma_a))} \quad (3.1.92)$$

or

$$\mu\alpha p^2 - \alpha(\delta + \mu^2\gamma)pq + \mu\alpha\gamma\delta q^2 \equiv 0 \pmod{(2a_1b_1c^2)(2b_1/(\gamma_2\gamma_b))} \quad (3.1.93)$$

and

$$\alpha(\delta - \mu^2\gamma)pq \equiv 0 \pmod{(2a_1b_1c^2)(2a_1/(\gamma_2\gamma_a))} \quad (3.1.94)$$

for the  $b$ -series (the sign  $-$ ).

Consider the condition (3.1.35) of Theorem 3.1.3. Consider the  $a$ -series (the sign  $+$ ). Second condition in (3.1.35) gives  $x \equiv \beta(2a_1b_1c/\gamma) \pmod{\delta}$ . By (3.1.66), we get  $(x, y) = \beta(2a_1b_1c/\gamma + \alpha\delta q^2, \alpha pq)$ . Then third condition in (3.1.35) is equivalent to

$$\beta\delta\alpha pq \equiv \mu(\beta 2a_1b_1c + \beta\alpha\gamma\delta q^2 - \beta 2a_1b_1c) \pmod{2a_1b_1c\delta}.$$

This is equivalent to  $\alpha q(p - \mu\gamma q) \equiv 0 \pmod{2a_1b_1c}$ . Since  $\gamma|2a_1b_1c$ , it follows the first condition in (3.1.35) which is  $c\delta\beta\alpha pq \equiv 0 \pmod{\gamma}$ .

For  $b$ -series we get the same. Thus, we have

**Proposition 3.1.9.** *The condition (3.1.35) of Theorem 3.1.3 is equivalent to*

$$\alpha q(p - \mu\gamma q) \equiv 0 \pmod{2a_1b_1c}. \quad (3.1.95)$$

Consider (3.1.34) of Theorem 3.1.3. We consider the  $a$ -series. Then

$$(x, y) = \beta\left(\frac{2a_1b_1c}{\gamma} + \alpha\delta q^2, \alpha pq\right) \quad (3.1.96)$$

The first relation of (3.1.34) gives

$$m(a, b)\gamma x + \mu\gamma y \equiv 2a_1b_1\beta c \pmod{2a_1b_1\gamma}.$$

Using (3.1.96), we get

$$m(a, b)\alpha\gamma\delta q^2 + \mu\alpha\gamma pq \equiv 2a_1b_1c(1 - m(a, b)) \pmod{2a_1b_1\gamma}.$$

This is equivalent to

$$-\alpha\delta q^2 + \mu\alpha pq \equiv \frac{4a_1b_1c}{\gamma} \pmod{2a_1}$$

and

$$\alpha\delta q^2 + \mu\alpha pq \equiv 0 \pmod{2b_1}.$$

Using (3.1.71), we can rewrite the first relation in the homogeneous form

$$\alpha p(p - \mu\gamma q) \equiv 0 \pmod{2a_1\gamma}.$$

The second relation of (3.1.34) gives

$$\delta m(a, b)y + \mu\gamma x - 2a_1b_1\beta\mu c \equiv 0 \pmod{2a_1b_1\delta}.$$

By (3.1.96) it is equivalent to

$$m(a, b)\alpha pq + \alpha\mu\gamma q^2 \equiv 0 \pmod{2a_1b_1}.$$

This is equivalent to

$$-\alpha pq + \alpha\mu\gamma q^2 \equiv 0 \pmod{2a_1}$$

and

$$\alpha pq + \alpha\mu\gamma q^2 \equiv 0 \pmod{2b_1}.$$

The third relation in (3.1.34) is

$$\begin{aligned} & \delta(m(a, b)\gamma x + \mu\gamma y - 2\beta a_1b_1c) \equiv \\ & \mu\gamma(\delta m(a, b)y + \mu\gamma x - 2\mu\beta a_1b_1c) \pmod{(2a_1b_1c^2\delta)(2a_1b_1)}. \end{aligned}$$

Using (3.1.96), one can calculate that it is equivalent to

$$\begin{aligned} & \mu\alpha\gamma pq(m(a, b) - 1) + \alpha\gamma q^2(\gamma\mu^2 - m(a, b)\delta) \\ & \equiv 2a_1b_1c(m(a, b) - 1) \pmod{(2a_1b_1c^2)(2a_1b_1)}. \end{aligned}$$

This is equivalent to

$$4a_1b_1c \equiv \alpha\gamma q(2\mu p - (\gamma\mu^2 + \delta)q) \pmod{(2a_1b_1c^2)(2a_1)}$$

and

$$\alpha\gamma q^2(\delta - \gamma\mu^2) \equiv 0 \pmod{(2a_1b_1c^2)(2b_1)}.$$

Using (3.1.71), we can rewrite the first relation in the homogeneous form

$$\alpha(p - \mu\gamma q)^2 \equiv 0 \pmod{(2a_1b_1c^2)(2a_1)}.$$

Thus, we get

**Proposition 3.1.10.** *The condition (3.1.34) of Theorem 3.1.3 is equivalent to the system of congruences:*

*For the a-series (the sign +)*

$$\alpha p(p - \mu\gamma q) \equiv 0 \pmod{2a_1\gamma}, \quad (3.1.97)$$

$$\alpha\delta q^2 + \alpha\mu pq \equiv 0 \pmod{2b_1}, \quad (3.1.98)$$

$$-\alpha pq + \alpha\mu\gamma q^2 \equiv 0 \pmod{2a_1}, \quad (3.1.99)$$

$$\alpha pq + \alpha\mu\gamma q^2 \equiv 0 \pmod{2b_1}, \quad (3.1.100)$$

$$\alpha(p - \mu\gamma q)^2 \equiv 0 \pmod{(2a_1b_1c^2)(2a_1)}, \quad (3.1.101)$$

$$\alpha\gamma q^2(\delta - \gamma\mu^2) \equiv 0 \pmod{(2a_1b_1c^2)(2b_1)}. \quad (3.1.102)$$

For the  $b$ -series (the sign  $-$ ):

$$\alpha p(p - \mu\gamma q) \equiv 0 \pmod{2b_1\gamma}, \quad (3.1.103)$$

$$\alpha\delta q^2 + \alpha\mu pq \equiv 0 \pmod{2a_1}, \quad (3.1.104)$$

$$-\alpha pq + \alpha\mu\gamma q^2 \equiv 0 \pmod{2b_1}, \quad (3.1.105)$$

$$\alpha pq + \alpha\mu\gamma q^2 \equiv 0 \pmod{2a_1}, \quad (3.1.106)$$

$$\alpha(p - \mu\gamma q)^2 \equiv 0 \pmod{(2a_1b_1c^2)(2b_1)}, \quad (3.1.107)$$

$$\alpha\gamma q^2(\delta - \gamma\mu^2) \equiv 0 \pmod{(2a_1b_1c^2)(2a_1)}. \quad (3.1.108)$$

Consider the condition (iv) of Theorem 3.1.3. It means that the corresponding element  $\tilde{h}$  is primitive. Since  $\tilde{h}^2 = 2a_1b_1$  and the lattice  $N(X)$  is even, it is not valid only if  $\tilde{h}/l \in N(X)$  for some prime  $l$  such that  $l^2|a_1b_1$ . Thus, (3.1.36) is not valid if and only if

$$x \equiv y \equiv \frac{x - \mu y}{2a_1b_1c^2/\gamma} \equiv 0 \pmod{l}$$

for some prime  $l$  such that  $l^2|a_1b_1$ . Using (3.1.66) and (3.1.71), we get

**Proposition 3.1.11.** *The condition (iv) of Theorem 3.1.3 is equivalent to the non-existence of a prime  $l$  such that  $l^2|a_1b_1$  and*

$$x \equiv y \equiv \frac{x - \mu y}{2a_1b_1c^2/\gamma} \equiv 0 \pmod{l}. \quad (3.1.109)$$

Equivalently, the system of congruences

$$\begin{cases} \alpha p^2 + \alpha\gamma\delta q^2 \equiv 0 \pmod{2\gamma l} \\ \alpha pq \equiv 0 \pmod{l} \\ \alpha p^2 + \alpha\gamma\delta q^2 \equiv 2\alpha\gamma\mu pq \pmod{4a_1b_1c^2l} \end{cases} \quad (3.1.110)$$

is not satisfied for any prime  $l$  such that  $l^2|a_1b_1$ .

Consider the condition (v) of Theorem 3.1.3. This is equivalent to  $\gamma(\tilde{h}) = \gamma$  where  $\tilde{h} \cdot N(X) = \gamma(\tilde{h})\mathbb{Z}$ . All other conditions of Theorem 3.1.3 give that  $\gamma|\tilde{h} \cdot N(X)$ , and  $\gamma|\gamma(\tilde{h})$ . Since  $\tilde{h}^2 = 2a_1b_1$ , it follows that  $(\gamma(\tilde{h})/\gamma) | 2a_1b_1/\gamma$ . Equivalently, (v) is not satisfied if and only if for some prime  $l|2a_1b_1/\gamma$  one has

$$x \equiv \frac{\delta y}{\gamma} \equiv \frac{\mu\gamma x - \delta y}{2a_1b_1c^2} \equiv 0 \pmod{l}.$$

Using (3.1.66) and (3.1.71), we get



**Proposition 3.1.12.** *The condition (v) of Theorem 3.1.3 is equivalent to the non-existence of a prime  $l \mid (2a_1b_1/\gamma)$  such that*

$$x \equiv \frac{\delta y}{\gamma} \equiv \frac{\mu\gamma x - \delta y}{2a_1b_1c^2} \equiv 0 \pmod{l}. \quad (3.1.111)$$

*Equivalently, the system of congruences*

$$\begin{cases} \alpha p^2 + \alpha\gamma\delta q^2 \equiv 0 \pmod{2\gamma l} \\ \delta\alpha pq \equiv 0 \pmod{\gamma l} \\ \mu\alpha p^2 + \mu\alpha\gamma\delta q^2 \equiv 2\alpha\delta pq \pmod{4a_1b_1c^2l} \end{cases} \quad (3.1.112)$$

*is not satisfied for any prime  $l \mid (2a_1b_1/\gamma)$ .*

Now we collect analysed conditions of Theorem 3.1.3 all together. We divide them in *general conditions*:  $(G')$  which are valid for both  $a$  and  $b$ -series, *conditions*  $(A')$  which are valid for the  $a$ -series, and *conditions*  $(B')$  which are valid for the  $b$ -series.

**$(G')$ : General conditions**

$$\alpha(p - \mu\gamma q)^2 \equiv 0 \pmod{4a_1b_1c^2}, \quad (3.1.113)$$

$$2\alpha pq(\delta - \gamma\mu^2) \equiv 0 \pmod{4a_1b_1c^2}, \quad (3.1.114)$$

$$\alpha q(p - \mu\gamma q) \equiv 0 \pmod{2a_1b_1c}, \quad (3.1.115)$$

$$\begin{cases} \alpha p^2 + \alpha\gamma\delta q^2 \equiv 0 \pmod{2\gamma l} \\ \alpha pq \equiv 0 \pmod{l} \\ \alpha p^2 + \alpha\gamma\delta q^2 \equiv 2\alpha\gamma\mu pq \pmod{4a_1b_1c^2l} \end{cases} \quad (3.1.116)$$

is not satisfied for any prime  $l$  such that  $l^2 \mid a_1b_1$ ,

$$\begin{cases} \alpha p^2 + \alpha\gamma\delta q^2 \equiv 0 \pmod{2\gamma l} \\ \delta\alpha pq \equiv 0 \pmod{\gamma l} \\ \mu\alpha p^2 + \mu\alpha\gamma\delta q^2 \equiv 2\alpha\delta pq \pmod{4a_1b_1c^2l} \end{cases} \quad (3.1.117)$$

is not satisfied for any prime  $l \mid (2a_1b_1/\gamma)$ .

**$(A')$ : Conditions of the  $a$ -series**

$$\alpha(\gamma_b q)^2 \equiv 0 \pmod{b_1}, \quad (3.1.118)$$

$$\mu\alpha p^2 - \alpha(\delta + \mu^2\gamma)pq + \mu\alpha\gamma\delta q^2 \equiv 0 \pmod{(2a_1b_1c^2)(2a_1/(\gamma_2\gamma_a))} \quad (3.1.119)$$

$$\alpha(\delta - \mu^2\gamma)pq \equiv 0 \pmod{(2a_1b_1c^2)(2b_1/(\gamma_2\gamma_b))}, \quad (3.1.120)$$

$$\alpha p(p - \mu\gamma q) \equiv 0 \pmod{2a_1\gamma}, \quad (3.1.121)$$

$$\alpha\delta q^2 + \alpha\mu pq \equiv 0 \pmod{2b_1}, \quad (3.1.122)$$

$$-\alpha pq + \alpha\mu\gamma q^2 \equiv 0 \pmod{2a_1}, \quad (3.1.123)$$

$$\alpha pq + \alpha\mu\gamma q^2 \equiv 0 \pmod{2b_1}, \quad (3.1.124)$$

$$\alpha(p - \mu\gamma q)^2 \equiv 0 \pmod{(2a_1b_1c^2)(2a_1)}, \quad (3.1.125)$$

$$\alpha\gamma q^2(\delta - \gamma\mu^2) \equiv 0 \pmod{(2a_1b_1c^2)(2b_1)}. \quad (3.1.126)$$

**(B'):** Conditions of the  $b$ -series

$$\alpha(\gamma_a q)^2 \equiv 0 \pmod{a_1}, \quad (3.1.127)$$

$$\mu\alpha p^2 - \alpha(\delta + \mu^2\gamma)pq + \mu\alpha\gamma\delta q^2 \equiv 0 \pmod{(2a_1b_1c^2)(2b_1/(\gamma_2\gamma_b))} \quad (3.1.128)$$

$$\alpha(\delta - \mu^2\gamma)pq \equiv 0 \pmod{(2a_1b_1c^2)(2a_1/(\gamma_2\gamma_a))}, \quad (3.1.129)$$

$$\alpha p(p - \mu\gamma q) \equiv 0 \pmod{2b_1\gamma}, \quad (3.1.130)$$

$$\alpha\delta q^2 + \alpha\mu pq \equiv 0 \pmod{2a_1}, \quad (3.1.131)$$

$$-\alpha pq + \alpha\mu\gamma q^2 \equiv 0 \pmod{2b_1}, \quad (3.1.132)$$

$$\alpha pq + \alpha\mu\gamma q^2 \equiv 0 \pmod{2a_1}, \quad (3.1.133)$$

$$\alpha(p - \mu\gamma q)^2 \equiv 0 \pmod{(2a_1b_1c^2)(2b_1)}, \quad (3.1.134)$$

$$\alpha\gamma q^2(\delta - \gamma\mu^2) \equiv 0 \pmod{(2a_1b_1c^2)(2a_1)}. \quad (3.1.135)$$

**3.2. Simplification of the conditions  $(G')$ ,  $(A')$  and  $(B')$ .** We have the following fundamental result which completely determines  $\alpha$  up to multiplication by  $\pm 1$ .

**Lemma 3.2.1.** *For the  $a$ -series, we have that the square-free  $\alpha|b_1$  and*

$$b_1/|\alpha| = \tilde{b}_1^2, \quad \tilde{b}_1 > 0,$$

*is a square.*

*Respectively, for the  $b$ -series, we have that the square-free  $\alpha|a_1$  and*

$$a_1/|\alpha| = \tilde{a}_1^2, \quad \tilde{a}_1 > 0,$$

*is a square.*

*Proof.* First let us prove that

$$\alpha|a_1b_1. \quad (3.2.1)$$

Otherwise,  $2a_1b_1/\alpha$  is odd. Let us consider the  $a$ -series. By (3.1.125), we have  $\alpha(p - \mu\gamma q)^2 \equiv 0 \pmod{4c^2}$ . Since  $\alpha$  is square-free, it follows that  $p - \mu\gamma q \equiv 0 \pmod{2c}$ . It follows  $p + \mu\gamma q \equiv 0 \pmod{2}$ . Then  $p^2 - \mu^2\gamma^2 \equiv 0 \pmod{4c}$ . We have  $\mu^2\gamma \equiv \delta \pmod{4a_1b_1c^2/\gamma}$ . Then  $\mu^2\gamma^2 \equiv \gamma\delta \pmod{4c^2}$ . Thus, we obtain  $p^2 - \gamma\delta q^2 \equiv 0$

mod  $4c$ . On the other hand,  $p^2 - \gamma\delta q^2 = 4a_1b_1c/\alpha \equiv 2c \pmod{4c}$  if  $2a_1b_1/\alpha$  is odd. We get a contradiction. It proves (3.2.1).

Now let us consider the  $a$ -series, and let us prove that

$$\alpha|b_1. \quad (3.2.2)$$

Otherwise, for a prime  $l$  one has  $l|\alpha$ ,  $l|a_1$ , but  $l$  does not divide  $b_1$ .

By (3.1.125), we have  $\alpha(p - \mu\gamma q)^2 \equiv 0 \pmod{4a_1^2c^2}$ . Since  $\alpha$  is square-free, it follows that  $p - \mu\gamma q \equiv 0 \pmod{2a_1c}$ . It follows  $p + \mu\gamma q \equiv 0 \pmod{2}$ . Then  $p^2 - \mu^2\gamma^2 \equiv 0 \pmod{4a_1c}$ . We have  $\mu^2\gamma \equiv \delta \pmod{4a_1b_1c^2/\gamma}$ . Then  $\mu^2\gamma^2 \equiv \gamma\delta \pmod{4a_1c^2}$ . Thus, we obtain  $p^2 - \gamma\delta q^2 \equiv 0 \pmod{4a_1c}$ . Thus, we obtain  $p^2 - \gamma\delta q^2 = 4a_1b_1c/\alpha \equiv 0 \pmod{4a_1c}$ . Equivalently,  $a_1b_1/\alpha \equiv 0 \pmod{a_1}$ . This gives a contradiction if for a prime  $l$  one has  $l|\alpha$ ,  $l|a_1$ , but  $l$  does not divide  $b_1$ . This proves (3.2.2).

Now let us prove that

$$b_1/|\alpha| \quad (3.2.3)$$

is a square. Otherwise, for a prime  $l$  we have  $l^{2t-1}|b_1/\alpha$ , but  $l^{2t}$  does not divide  $b_1/\alpha$  where  $t \geq 1$ . By (3.1.118),  $(\gamma q)^2 \equiv 0 \pmod{b_1/\alpha}$ . It follows that

$$l^t|\gamma q. \quad (3.2.4)$$

By (3.1.113), we have

$$p - \gamma\mu q \equiv 0 \pmod{l^t}. \quad (3.2.5)$$

By (3.2.4) and (3.1.5), we obtain  $l^t|p$ .

From  $p^2 - \gamma\delta q^2 = 4a_1(b_1/\alpha)c$  we then get a contradiction if  $l$  does not divide  $2c$ .

We have

$$\gamma\delta \equiv \gamma^2\mu^2 \pmod{4a_1b_1c^2}. \quad (3.2.6)$$

From  $p^2 - \gamma\delta q^2 = 4a_1(b_1/\alpha)c$  and (3.2.6) we then get

$$p^2 - \gamma^2\mu^2q^2 \equiv (p - \gamma\mu q)(p + \gamma\mu q) \equiv 4a_1(b_1/\alpha)c \pmod{4a_1b_1c^2}.$$

Then, using (3.2.5), we get

$$\frac{p - \gamma\mu q}{2a_1cl^t}(p + \gamma\mu q) \equiv 2(b_1/\alpha)/l^t \pmod{2(b_1/l^t)c} \quad (3.2.7)$$

where  $(p - \gamma\mu q)/(2a_1cl^t)$  is an integer.

If  $l|c$  and  $l$  is odd, (3.2.5) and (3.2.7) give a contradiction because  $l^t|p + \gamma\mu q$ ,  $l^t|2(b_1/l^t)c$ , but  $l^t$  does not divide  $2(b_1/\alpha)/l^t$ .

Now assume that  $l = 2|c$ . If  $2^{t+1}|\gamma\mu q$ , then  $2^{t+1}|p$  by (3.2.5), and we get a contradiction in the same way. Assume that  $2^{t+1}$  does not divide  $\gamma\mu q$ . Then  $\gamma\mu q/2^t$  and  $p/2^t$  are both odd, and  $2^{t+1}|p + \gamma\mu q$ . It also leads to a contradiction in the same way.

Now assume that  $l = 2$  and  $c$  is odd. By (3.1.126), we obtain

$$\gamma\delta q^2 \equiv (\gamma\mu q)^2 \pmod{2^{4t}}. \quad (3.2.8)$$

Assume that  $2^{t+1} \mid \mu\gamma q$ . By (3.2.8), then  $\gamma\delta q^2 \equiv 0 \pmod{2^{2t+2}}$ . By (3.2.5),  $2^{t+1} \mid p$  and  $2^{2t+2} \mid p^2$ . Then  $2^{2t+1} \mid p^2 - \gamma\delta q^2 = 4a_1(b_1/\alpha)c$  which gives a contradiction because  $4a_1(b_1/\alpha)c$  is divisible by  $2^{2t+1}$  only.

Now assume that  $2^{t+1}$  does not divide  $\mu\gamma q$ . By (3.2.8), we get  $\gamma\delta q^2 \equiv 2^{2t} \pmod{2^{2t+2}}$ . By (3.2.5) then  $2^t \mid p$ , but  $2^{t+1}$  does not divide  $p$ . It follows that  $p^2 \equiv 2^{2t} \pmod{2^{2t+2}}$ . Then  $2^{2t+2} \mid p^2 - \gamma\delta q^2 = 4a_1(b_1/\alpha)c$  which again leads to a contradiction.

This finishes the proof of the theorem.

**Lemma 3.2.2.** *Assume that*

$$g.c.d(\delta - \gamma\mu^2, 4a_1b_1c^2) = (4a_1b_1c^2/\gamma)\gamma_0$$

where  $\gamma_0 \mid \gamma$ .

For any  $u \mid a_1b_1c^2/(\gamma_a\gamma_b)$  and  $g.c.d(u, \gamma/\gamma_0) = 1$  we can choose

$$\mu \in (\mathbb{Z}/((2a_1b_1c^2/\gamma)\gamma_0u))^*$$

such that

$$\delta \equiv \gamma\mu^2 \pmod{(4a_1b_1c^2/\gamma)\gamma_0u}.$$

*Proof.* Assume  $\mu_0 \pmod{2a_1b_1c^2/\gamma} \in (\mathbb{Z}/(2a_1b_1c^2/\gamma))^*$  and

$$\delta \equiv \gamma\mu_0^2 \pmod{4a_1b_1c^2/\gamma}.$$

Taking  $\mu = \mu_0 + (2a_1b_1c^2/\gamma)k$ , we get

$$\delta - \gamma\mu^2 = \delta - \gamma\mu_0^2 - 4\mu_0a_1b_1c^2k - (4a_1^2b_1^2c^4/\gamma)k^2.$$

Then

$$(\delta - \gamma\mu^2)/(4a_1b_1c^2/\gamma) \equiv (\delta - \gamma\mu_0^2)/(4a_1b_1c^2/\gamma) + \gamma\mu_0k \pmod{\gamma_0u}.$$

Since  $(\gamma/\gamma_0)\mu_0$  are invertible  $\pmod{u}$ , we can choose  $k$  such that

$$(\delta - \gamma\mu^2)/(4a_1b_1c^2/\gamma) \equiv 0 \pmod{\gamma_0u}.$$

It follows the statement.

Further we consider the  $a$ -series (similar results will be valid for  $b$ -series).

The congruence (3.1.125) implies (3.1.113). The (3.1.125) is equivalent to

$$p - \mu\gamma q \equiv 0 \pmod{2a_1\tilde{b}_1c}. \quad (3.2.9)$$

Thus, (3.1.113) and (3.1.125) are equivalent to (3.2.9).

The congruence (3.1.118) is equivalent to  $\gamma_b q \equiv 0 \pmod{\tilde{b}_1}$ . Equivalently,

$$q = \frac{\tilde{b}_1 q_1}{\gamma_b}, \quad \tilde{b}_1 q_1 \equiv 0 \pmod{\gamma_b} \quad (3.2.10)$$

where  $q_1$  is an integer.

By (3.2.9), we have  $p - \mu\gamma_2\gamma_a\tilde{b}_1q_1 \equiv 0 \pmod{2a_1\tilde{b}_1c}$ , and then  $\gamma_2\gamma_a\tilde{b}_1|p$ . Then

$$p = \gamma_2\gamma_a\tilde{b}_1p_1 \quad (3.2.11)$$

where  $p_1$  is an integer. The congruence (3.2.9) is then equivalent to

$$p_1 - \mu q_1 \equiv 0 \pmod{(2/\gamma_2)(a_1/\gamma_a)c}. \quad (3.2.12)$$

Denoting

$$\alpha = \pm b_1/\tilde{b}_1^2, \quad (3.2.13)$$

we can rewrite  $p^2 - \gamma\delta q^2 = 4a_1b_1c/\alpha$  as

$$\gamma p_1^2 - \delta q_1^2 = \pm 2(2/\gamma_2)(a_1/\gamma_a)\gamma_b c. \quad (3.2.14)$$

Now we can rewrite conditions  $(G')$ ,  $(A')$  and  $(B')$  using the introduced  $(p_1, q_1)$ .

As we have seen, the conditions (3.1.113), (3.1.118) and (3.1.125) are equivalent to (3.2.12) and

$$\tilde{b}_1 q_1 \equiv 0 \pmod{\gamma_b}. \quad (3.2.15)$$

The condition (3.1.114) gives

$$p_1 q_1 (\delta - \gamma\mu^2) \equiv 0 \pmod{(2/\gamma_2)(a_1/\gamma_a)c^2\gamma_b}.$$

Since  $\text{g.c.d}(\gamma, c) = \text{g.c.d}(\gamma_b, a_1c^2) = 1$  and  $\delta - \gamma\mu^2 \equiv 0 \pmod{4a_1b_1c^2/\gamma}$ , it is equivalent to

$$p_1 q_1 (\delta - \gamma\mu^2) \equiv 0 \pmod{(2/\gamma_2)\gamma_b}. \quad (3.2.16)$$

The condition (3.1.115) gives

$$q_1(p_1 - \mu q_1) \equiv 0 \pmod{(2/\gamma_2)(a_1/\gamma_a)c\gamma_b}.$$

Since  $\text{g.c.d}(\gamma, c) = \text{g.c.d}(\gamma_b, a_1c^2) = 1$  and  $p_1 - \mu q_1 \equiv 0 \pmod{(2/\gamma_2)(a_1/\gamma_a)c}$  (by (3.2.12)), it is similarly equivalent to

$$q_1(p_1 - \mu q_1) \equiv 0 \pmod{(2/\gamma_2)\gamma_b}. \quad (3.2.17)$$

The condition (3.1.120) gives

$$(\delta - \gamma\mu^2)p_1 q_1 \equiv 0 \pmod{(4/\gamma_2^2)b_1c^2}. \quad (3.2.18)$$

Since  $(\delta - \mu^2\gamma) \equiv 0 \pmod{4b_1c^2/(\gamma_2\gamma_b)}$ , the congruence (3.2.18) is actually a congruence  $\pmod{\gamma_b}$  on  $p_1q_1$ . It also implies (3.2.16).

The condition (3.1.121) gives  $b_1p_1(p_1 - \mu q_1) \equiv 0 \pmod{(2/\gamma_2)(a_1/\gamma_a)\gamma_b}$ . It satisfies because of (3.2.12) and  $\gamma_b|b_1$ .

The condition (3.1.122) gives

$$q_1(\delta q_1 + \mu\gamma p_1) \equiv 0 \pmod{2\gamma_b^2}. \quad (3.2.19)$$

The condition (3.1.123) gives  $(b_1/\gamma_b)q_1(-p_1 + \mu q_1) \equiv 0 \pmod{(2/\gamma_2)(a_1/\gamma_a)}$ . It satisfies by (3.2.12).

The condition (3.1.124) gives

$$\gamma_a q_1(p_1 + \mu q_1) \equiv 0 \pmod{(2/\gamma_2)\gamma_b}. \quad (3.2.20)$$

It is easy to see that (3.2.17), (3.2.20) together with  $\delta \equiv \mu^2\gamma \pmod{4a_1b_1c^2/\gamma}$  imply (3.2.18).

Taking  $\pm\gamma_a$  (3.2.17) plus (3.2.20), we obtain

$$\gamma_2 p_1 q_1 \equiv 0 \pmod{\gamma_b} \quad (3.2.21)$$

and  $\gamma_2 \mu q_1^2 \equiv 0 \pmod{\gamma_b}$ . Since  $\mu$  can be always taken coprime to  $\gamma_b$ , the last congruence is equivalent to

$$\gamma_2 q_1^2 \equiv 0 \pmod{\gamma_b}. \quad (3.2.22)$$

Any of them together with (3.2.17) can be taken to replace (3.2.20).

The condition (3.1.126) is equivalent to

$$(\delta - \gamma\mu^2)q_1^2 \equiv 0 \pmod{(4/\gamma_2)b_1c^2\gamma_b}. \quad (3.2.23)$$

By (3.2.22) and  $(\delta - \mu^2\gamma) \equiv 0 \pmod{4b_1c^2/(\gamma_2\gamma_b)}$ , the congruence (3.2.23) is actually a congruence  $\pmod{\gamma_b}$ .

It is easy to see that (3.2.23) and (3.2.20) imply (3.2.19).

The condition (3.1.119) gives

$$\mu\gamma p_1^2 - (\delta + \mu^2\gamma)p_1q_1 + \mu\delta q_1^2 \equiv 0 \pmod{(4/\gamma_2^2)(a_1^2/\gamma_a^2)\gamma_b c^2}.$$

Since  $a_1$  and  $b_1$  are coprime, this is equivalent to two congruences

$$\mu\gamma p_1^2 - (\delta + \mu^2\gamma)p_1q_1 + \mu\delta q_1^2 \equiv 0 \pmod{(4/\gamma_2^2)c^2\gamma_b}$$

and

$$\mu\gamma p_1^2 - (\delta + \mu^2\gamma)p_1q_1 + \mu\delta q_1^2 \equiv 0 \pmod{(4/\gamma_2^2)c^2(a_1^2/\gamma_a^2)}. \quad (3.2.24)$$

By (3.2.23), we have  $\delta q_1^2 \equiv \gamma\mu^2 q_1^2 \pmod{(4/\gamma_2^2)c^2\gamma_b}$ , and the first congruence gives

$$\mu\gamma(p_1 - \mu q_1)^2 - (\delta - \mu^2\gamma)p_1q_1 \equiv 0 \pmod{(4/\gamma_2^2)c^2\gamma_b}$$

It satisfies because of (3.2.12) and (3.2.18). The congruence (3.2.24) can be written as

$$\mu\gamma(p_1 - \mu q_1)^2 + (\delta - \mu^2\gamma)(p_1 - \mu q_1)q_1 \equiv 0 \pmod{(4/\gamma_2^2)c^2(a_1^2/\gamma_a^2)}.$$

It satisfies because of (3.2.12) and since  $\delta - \gamma\mu^2 \equiv 0 \pmod{(4/\gamma_2)(a_1/\gamma_a)c^2}$ .

The condition (3.1.116) is equivalent to

$$\begin{cases} (b_1/\gamma_b)(\gamma p_1^2 + \delta q_1^2) \equiv 0 \pmod{2\gamma_b l} \\ (b_1/\gamma_b)\gamma_2\gamma_a p_1 q_1 \equiv 0 \pmod{l} \\ \gamma^2 p_1^2 + \gamma\delta q_1^2 \equiv 2\gamma^2 \mu p_1 q_1 \pmod{4a_1 c^2 \gamma_b^2 l} \end{cases} \quad (3.2.25)$$

is not satisfied for any prime  $l$  such that  $l^2 | a_1 b_1$ .

By its meaning, the congruences (3.2.25) satisfies if we formally put  $l = 1$ . Assume that for a prime  $l$  we have  $l^2 | a_1 b_1$  and  $\text{g.c.d}(l, \gamma) = 1$ . Then (3.2.25) is equivalent to

$$\begin{cases} b_1(\gamma p_1^2 + \delta q_1^2) \equiv 0 \pmod{2l} \\ b_1 p_1 q_1 \equiv 0 \pmod{l} \\ \gamma p_1^2 + \delta q_1^2 \equiv 2\gamma \mu p_1 q_1 \pmod{(4/\gamma_2)(a_1/\gamma_a)c^2 l} \end{cases}$$

is not satisfied. By Lemma 3.2.2, we can assume that

$\delta \equiv \gamma\mu^2 \pmod{(4/\gamma_2)(a_1/\gamma_a)lc^2}$ , and the last condition is equivalent to

$$\begin{cases} b_1(\gamma p_1^2 + \delta q_1^2) \equiv 0 \pmod{2l} \\ b_1 p_1 q_1 \equiv 0 \pmod{l} \\ \gamma(p_1 - \mu q_1)^2 \equiv 0 \pmod{(4/\gamma_2)(a_1/\gamma_a)lc^2} \end{cases}$$

is not satisfied. By (3.2.12), this is equivalent to

$$\begin{cases} b_1(\gamma p_1^2 + \delta q_1^2) \equiv 0 \pmod{2l} \\ b_1 p_1 q_1 \equiv 0 \pmod{l} \\ \gamma_a[(p_1 - \mu q_1)/(2c/\gamma_2)]^2 \equiv 0 \pmod{(a_1/\gamma_a)l} \end{cases} \quad (3.2.26)$$

is not satisfied. Assume that  $l | b_1$ . By (3.2.14), then the first and second congruences satisfy and (3.2.26) is equivalent to

$$p_1 - \mu q_1 \equiv 0 \pmod{(2/\gamma_2)(a_1/\gamma_a)cl} \quad (3.2.27)$$

does not satisfy. Assume that  $l | a_1$ . By (3.2.12) then third congruence in (3.2.26) satisfies, and by (3.2.12) and (3.2.14) the condition (3.2.26) is equivalent to

$$l \nmid p_1. \quad (3.2.28)$$

The condition (3.1.117) is equivalent to

$$\begin{cases} (b_1/\gamma_b)(\gamma p_1^2 + \delta q_1^2) \equiv 0 \pmod{2\gamma_b l} \\ (b_1/\gamma_b)p_1 q_1 \equiv 0 \pmod{\gamma_b l} \\ \mu\gamma^2 p_1^2 + \mu\gamma\delta q_1^2 \equiv 2\delta\gamma p_1 q_1 \pmod{4a_1 c^2 \gamma_b^2 l} \end{cases} \quad (3.2.29)$$

is not satisfied for any prime  $l \mid (2a_1 b_1/\gamma)$ .

By its meaning, (3.2.29) satisfies if we formally put  $l = 1$ . Assume that  $\text{g.c.d}(l, \gamma) = 1$  and  $l \mid 2a_1 b_1/\gamma$ . Then (3.2.29) is equivalent to

$$\begin{cases} b_1(\gamma p_1^2 + \delta q_1^2) \equiv 0 \pmod{2l} \\ b_1 p_1 q_1 \equiv 0 \pmod{l} \\ \mu\gamma p_1^2 + \mu\delta q_1^2 \equiv 2\delta p_1 q_1 \pmod{(4/\gamma_2)(a_1/\gamma_2)c^2 l} \end{cases}$$

is not satisfied.

First assume that  $l \mid a_1 b_1$ . By Lemma 3.2.2, we can assume that  $\delta \equiv \gamma\mu^2 \pmod{(4/\gamma_2)(a_1/\gamma_a)lc^2}$ , and the last condition is equivalent to

$$\begin{cases} b_1(\gamma p_1^2 + \delta q_1^2) \equiv 0 \pmod{2l} \\ b_1 p_1 q_1 \equiv 0 \pmod{l} \\ \mu\gamma(p_1 - \mu q_1)^2 \equiv 0 \pmod{(4/\gamma_2)(a_1/\gamma_a)lc^2} \end{cases}$$

is not satisfied. We obtain exactly the same conditions (3.2.27) and (3.2.28) as above.

Now assume that  $l \nmid a_1 b_1$ . Then  $l = 2$  and  $a_1, b_1, \gamma, \delta$  are odd. We then obtain that

$$\begin{cases} \gamma p_1^2 + \delta q_1^2 \equiv 0 \pmod{2l} \\ p_1 q_1 \equiv 0 \pmod{l} \\ \mu\gamma p_1^2 + \mu\delta q_1^2 \equiv 2\delta p_1 q_1 \pmod{(4/\gamma_2)(a_1/\gamma_2)c^2 l} \end{cases} \quad (3.2.30)$$

is not satisfied. From first two congruences we get that  $p_1 \equiv q_1 \equiv 0 \pmod{2}$ . Then we get  $q_1 \delta \equiv q_1 \gamma \mu^2 \pmod{(4/\gamma_2)(a_1/\gamma_a)lc^2}$ , and we can rewrite (3.2.30) again as

$$\begin{cases} \gamma p_1^2 + \delta q_1^2 \equiv 0 \pmod{2l} \\ b_1 p_1 q_1 \equiv 0 \pmod{l} \\ \mu\gamma(p_1 - \mu q_1)^2 \equiv 0 \pmod{(4/\gamma_2)(a_1/\gamma_a)lc^2} \end{cases} \quad (3.2.31)$$

is not satisfied. From the last congruence in (3.2.31) we get  $\gamma(p_1 - \mu q_1) \equiv 0 \pmod{4c}$  and  $p_1 + \mu q_1 \equiv 0 \pmod{2}$ . It follows that  $\gamma p_1^2 - \gamma \mu^2 q_1^2 \equiv 0 \pmod{8c}$ . Moreover, we have  $q_1 \delta \equiv q_1 \gamma \mu^2 \pmod{8c}$ . Thus, we obtain  $\gamma p_1^2 - \delta q_1^2 \equiv 0 \pmod{8c}$ . It leads to a contradiction since  $\gamma p_1^2 - \delta q_1^2 = \pm 2(2/\gamma_2)(a_1/\gamma_a)\gamma_b c \equiv 4c \pmod{8c}$ .



In fact, (3.2.29) always follows from (3.2.29) if  $l \nmid 2a_1b_1$  and  $\text{g.c.d}(l, \gamma) = 1$ . Really, by our construction, (3.2.29) means that the corresponding element  $\tilde{h} \in N(X)$  of Theorem 3.1.3 is not divisible by  $l$ . We have  $\mu \in (\mathbb{Z}/(2a_1b_1c^2/\gamma))^*$ ,  $\delta \equiv \gamma\mu^2 \pmod{4a_1b_1c^2/\gamma}$ , and  $\det N(X) = -\gamma\delta$  is not divisible by  $l$ . Then  $N(X) \cdot \tilde{h}$  is not divisible by  $l$ , and (3.2.29) follows. In particular, we can assume that  $l^2 \nmid a_1b_1$ .

Thus, we can rewrite the conditions  $(G')$  and  $(A')$  respectively in the form

**(A): The conditions of  $a$ -series:**

**(AG): The general conditions of  $a$ -series:**

$$p_1 - \mu q_1 \equiv 0 \pmod{(2/\gamma_2)(a_1/\gamma_a)c}, \quad (3.2.32)$$

$$\begin{aligned} p_1 - \mu q_1 &\not\equiv 0 \pmod{(2/\gamma_2)(a_1/\gamma_a)cl} \\ \text{for any prime } l &\text{ such that } l^2 \nmid b_1 \text{ and } \text{g.c.d}(l, \gamma) = 1, \end{aligned} \quad (3.2.33)$$

$$l \nmid p_1 \text{ for any prime } l \text{ such that } l^2 \nmid a_1 \text{ and } \text{g.c.d}(l, \gamma) = 1. \quad (3.2.34)$$

**(AS) The singular conditions of  $a$ -series:**

$$\tilde{b}_1 q_1 \equiv 0 \pmod{\gamma_b}, \quad (3.2.35)$$

$$q_1(p_1 - \mu q_1) \equiv 0 \pmod{(2/\gamma_2)\gamma_b}, \quad (3.2.36)$$

$$\gamma_2 p_1 q_1 \equiv \gamma_2 q_1^2 \equiv 0 \pmod{\gamma_b}, \quad (3.2.37)$$

$$(\delta - \gamma\mu^2)q_1^2 \equiv 0 \pmod{(4/\gamma_2)b_1c^2\gamma_b}. \quad (3.2.38)$$

$$\begin{cases} (b_1/\gamma_b)(\gamma p_1^2 + \delta q_1^2) \equiv 0 \pmod{2\gamma_b l} \\ (b_1/\gamma_b)\gamma_2 \gamma_a p_1 q_1 \equiv 0 \pmod{l} \\ \gamma^2 p_1^2 + \gamma \delta q_1^2 \equiv 2\gamma^2 \mu p_1 q_1 \pmod{4a_1c^2\gamma_b^2 l} \end{cases} \quad (3.2.39)$$

is not satisfied for any prime  $l$  such that  $l^2 \nmid a_1b_1$  and  $l \nmid \gamma$ ,

$$\begin{cases} (b_1/\gamma_b)(\gamma p_1^2 + \delta q_1^2) \equiv 0 \pmod{2\gamma_b l} \\ (b_1/\gamma_b)p_1 q_1 \equiv 0 \pmod{\gamma_b l} \\ \mu \gamma^2 p_1^2 + \mu \gamma \delta q_1^2 \equiv 2\delta \gamma p_1 q_1 \pmod{4a_1c^2\gamma_b^2 l} \end{cases} \quad (3.2.40)$$

is not satisfied for any prime  $l \mid (2a_1b_1/\gamma)$  and  $l \nmid \gamma$ .

Similarly we can rewrite the conditions  $(G')$  and  $(B')$  respectively in the form

**(B): The conditions of  $b$ -series:**

**(BG): The general conditions of  $b$ -series:**

$$p_1 - \mu q_1 \equiv 0 \pmod{(2/\gamma_2)(b_1/\gamma_b)c}, \quad (3.2.41)$$

$$p_1 - \mu q_1 \not\equiv 0 \pmod{(2/\gamma_2)(b_1/\gamma_b)c l}$$

for any prime  $l$  such that  $l^2|a_1$  and  $\text{g.c.d}(l, \gamma) = 1$ ,

(3.2.42)

$$l \nmid p_1 \text{ for any prime } l \text{ such that } l^2|b_1 \text{ and } \text{g.c.d}(l, \gamma) = 1.$$
(3.2.43)

**(BS) The singular conditions of  $b$ -series:**

$$\tilde{a}_1 q_1 \equiv 0 \pmod{\gamma_a},$$
(3.2.44)

$$q_1(p_1 - \mu q_1) \equiv 0 \pmod{(2/\gamma_2)\gamma_a},$$
(3.2.45)

$$\gamma_2 p_1 q_1 \equiv \gamma_2 q_1^2 \equiv 0 \pmod{\gamma_a},$$
(3.2.46)

$$(\delta - \gamma\mu^2)q_1^2 \equiv 0 \pmod{(4/\gamma_2)a_1 c^2 \gamma_a}.$$
(3.2.47)

$$\begin{cases} (a_1/\gamma_a)(\gamma p_1^2 + \delta q_1^2) \equiv 0 \pmod{2\gamma_a l} \\ (a_1/\gamma_a)\gamma_2 \gamma_b p_1 q_1 \equiv 0 \pmod{l} \\ \gamma^2 p_1^2 + \gamma \delta q_1^2 \equiv 2\gamma^2 \mu p_1 q_1 \pmod{4b_1 c^2 \gamma_a^2 l} \end{cases}$$
(3.2.48)

is not satisfied for any prime  $l$  such that  $l^2|a_1 b_1$  and  $l|\gamma$ ,

$$\begin{cases} (a_1/\gamma_a)(\gamma p_1^2 + \delta q_1^2) \equiv 0 \pmod{2\gamma_a l} \\ (a_1/\gamma_a)p_1 q_1 \equiv 0 \pmod{\gamma_a l} \\ \mu \gamma^2 p_1^2 + \mu \gamma \delta q_1^2 \equiv 2\delta \gamma p_1 q_1 \pmod{4b_1 c^2 \gamma_a^2 l} \end{cases}$$
(3.2.49)

is not satisfied for any prime  $l \mid (2a_1 b_1/\gamma)$  and  $l|\gamma$ .

We remind that here

$$\mu \in (\mathbb{Z}/2a_1 b_1 c^2/\gamma)^*,$$
(3.2.50)

$$\delta \equiv \gamma\mu^2 \pmod{4a_1 b_1 c^2/\gamma},$$
(3.2.51)

$$\gamma p_1^2 - \delta q_1^2 = \pm 2(2/\gamma_2)(a_1/\gamma_a)\gamma_b c$$
(3.2.52)

for the  $a$ -series, and

$$\gamma p_1^2 - \delta q_1^2 = \pm 2(2/\gamma_2)(b_1/\gamma_b)\gamma_a c$$
(3.2.53)

for the  $b$ -series.

**3.3. The resolution of the singular conditions (AS) and (BS).** Here we completely resolve the singular conditions (AS) and (BS) (assuming the corresponding general conditions (3.2.50), (3.2.51), (3.2.52), (3.2.53), (3.2.32), (3.2.41) of these series). It makes all our results very effective.

Below we consider the singular condition (AS) of  $a$ -series. The singular condition (AS) consists of congruences and non-congruences  $\pmod{\gamma}$ . We denote by  $(\text{AS})^{(p)}$  the corresponding conditions over the prime number  $p|\gamma$ . It is enough to satisfy all conditions  $(\text{AS})^{(p)}$  for all  $p|\gamma$ . Below we consider several cases which all together cover all possible ones. For a prime  $p$  and a natural number  $n$ , we denote as  $n^{(p)} = p^{\nu_p} n$  the  $p$ -component of  $n$ . That is  $n^{(p)} = p^{\nu_p(n)}|n$ , and  $\text{g.c.d}(n^{(p)}, n/n^{(p)}) = 1$ .

There are several cases which we consider below.

*Case  $\gamma_2 = 2$ ,  $\gamma_a \equiv \gamma_b \equiv 1 \pmod{2}$ .* Then  $c$ ,  $a_1$ ,  $b_1$  are odd and  $(AS)^{(2)}$  obviously satisfies.

*Case  $\gamma_2 = 2$ ,  $2|\gamma_a$ .* Then  $\gamma^{(2)} = 2a_1^{(2)} \geq 4$ ,  $\gamma_b^{(2)} = 1$  and  $b_1 \equiv 1 \pmod{2}$ . By (3.2.51), we obtain that

$$\delta \equiv 0 \pmod{2}. \quad (3.3.1)$$

All conditions (AS) trivially satisfy over 2 except (3.2.39) for  $l = 2$  which gives  $\delta q_1^2 \not\equiv 0 \pmod{4}$  if  $4|a_1$ . It is equivalent to  $\delta \equiv 2 \pmod{4}$  and  $q_1 \equiv 1 \pmod{2}$  if  $4|a_1$ . This follows from (3.2.52) since  $4|\gamma$ . Thus (AS) over 2 satisfies.

*Case  $\gamma_2 = 2$ ,  $2|\gamma_b$ .* Then  $\gamma_a^{(2)} = 1$  and  $\gamma^{(2)} = 2\gamma_b^{(2)} \geq 4$ ,  $b_1^{(2)} = \gamma_b^{(2)}$ . Denote  $\gamma^{(2)} = 2^t$ ,  $t \geq 2$ . For  $l = 2$ , the condition (3.2.40) satisfies, and (3.2.35)—(3.2.39) give over 2 respectively

$$q_1 \equiv 0 \pmod{2^{[t/2]}}, \quad (3.3.2)$$

$$q_1(p_1 - \mu q_1) \equiv 0 \pmod{2^{t-1}}, \quad (3.3.3)$$

$$p_1 q_1 \equiv q_1^2 \equiv 0 \pmod{2^{t-2}}, \quad (3.3.4)$$

$$(\delta - \gamma \mu^2) q_1^2 \equiv 0 \pmod{2^{2t-1}}, \quad (3.3.5)$$

$$\gamma p_1^2 + \delta q_1^2 \not\equiv 0 \pmod{2^{t+1}} \text{ if } t \geq 3. \quad (3.3.6)$$

By (3.2.51),  $\delta$  is even.

Assume  $t = 2$ . Then (3.3.2)—(3.3.6) are equivalent to  $q_1 \equiv 0 \pmod{2}$ . By (3.2.52), then  $p_1 \equiv 1 \pmod{2}$ .

Assume that  $t$  is even and  $t \geq 4$ . By (3.3.2), we have  $q_1 \equiv 0 \pmod{2^{t/2}}$ . Then (3.3.6) is equivalent to  $p_1 \equiv 1 \pmod{2}$ . By (3.3.3), then  $q_1 \equiv 0 \pmod{2^{t-1}}$ . It follows (3.3.5). Thus, (3.3.2)—(3.3.6) are equivalent to  $p_1 \equiv 1 \pmod{2}$  and  $q_1 \equiv 0 \pmod{2^{t-1}}$ . We had the same for  $t = 2$ .

Assume that  $t$  is odd and  $t \geq 3$ . Let us suppose that  $p_1 \equiv 0 \pmod{2}$ . Then (3.3.6) gives  $\delta q_1^2 \not\equiv 0 \pmod{2^{t+1}}$ . By (3.3.2), we have  $q_1 \equiv 0 \pmod{2^{(t-1)/2}}$ . Moreover  $\delta$  is even. Then (3.3.6) is equivalent to  $\delta \equiv 2 \pmod{4}$  and  $q_1 \equiv 2^{(t-1)/2} \pmod{2^{(t-1)/2+1}}$ . Then  $(\delta - \gamma \mu^2) q_1^2 \equiv 2^t \pmod{2^{t+1}}$  and (3.3.5) is not valid. This shows that  $p_1 \equiv 1 \pmod{2}$ . By (3.3.2),  $q_1$  is even. By (3.3.3), then  $q_1 \equiv 0 \pmod{2^{t-1}}$ . These imply all conditions (3.3.2)—(3.3.6). Thus, we obtain the same conditions  $p_1$  is odd and  $q_1 \equiv 0 \pmod{2^{t-1}}$ . Thus, in this case, the condition (AS) over 2 is

$$\text{If } \gamma_2 = 2 \text{ and } 2|b_1, \text{ then } p_1 \equiv 1 \pmod{2} \text{ and } q_1 \equiv 0 \pmod{\gamma^{(2)}/2}. \quad (3.3.7)$$

*Case  $\gamma_2 = 1$ ,  $2|\gamma$ .*

*Case  $\gamma_2 = 1$ ,  $2|\gamma$  and  $2|a_1$ .* Then  $2 \leq \gamma^{(2)} = \gamma_a^{(2)}|a_1$ ,  $\gamma_b^{(2)} = 1$  and  $b_1 \equiv 1 \pmod{2}$ . By (3.2.50), (3.2.51) and (3.2.32), we get respectively

$$\mu \equiv 1 \pmod{2}, \quad (3.3.8)$$

$$\delta \equiv \gamma\mu^2 \pmod{4(a_1^{(2)}/\gamma_a^{(2)})}, \quad (3.3.9)$$

and then  $\delta \equiv \gamma \pmod{4}$ ,

$$p_1 - \mu q_1 \equiv 0 \pmod{2(a_1^{(2)}/\gamma_a^{(2)})}. \quad (3.3.10)$$

It follows that over 2 all conditions (AS) satisfy except (3.2.39) and (3.2.40) which give respectively

$$\gamma p_1^2 + \delta q_1^2 \equiv 2\gamma\mu p_1 q_1 \pmod{8(a_1^{(2)}/\gamma_a^{(2)})} \quad (3.3.11)$$

is not satisfied if  $4|a_1$ , and

$$\begin{cases} p_1 q_1 \equiv 0 \pmod{2} \\ \mu\gamma p_1^2 + \mu\delta q_1^2 \equiv 2\delta p_1 q_1 \pmod{8(a_1^{(2)}/\gamma_a^{(2)})} \end{cases} \quad (3.3.12)$$

is not satisfied.

By (3.3.10), we have  $\gamma p_1^2 + \gamma\mu q_1^2 \equiv 2\gamma\mu p_1 q_1 \pmod{4a_1(a_1^{(2)}/\gamma_a^{(2)})}$ . Since  $a_1$  is even, it follows

$$\gamma p_1^2 + \gamma\mu q_1^2 \equiv 2\gamma\mu p_1 q_1 \pmod{8(a_1^{(2)}/\gamma_a^{(2)})}. \quad (3.3.13)$$

Then (3.3.11) is equivalent to  $(\delta - \gamma\mu^2)q_1^2 \not\equiv 0 \pmod{8(a_1^{(2)}/\gamma_a^{(2)})}$  if  $4|a_1$ . By (3.3.9), this is equivalent to

$$q_1 \equiv 1 \pmod{2} \text{ and } \delta - \gamma\mu^2 \not\equiv 0 \pmod{8(a_1^{(2)}/\gamma_a^{(2)})} \quad (3.3.14)$$

if  $4|a_1$ .

Similarly, one can see that (3.3.12) is equivalent to

$$\begin{cases} p_1 q_1 \equiv 0 \pmod{2} \\ (\delta - \gamma\mu^2)q_1^2 \equiv 0 \pmod{8(a_1^{(2)}/\gamma_a^{(2)})} \end{cases}$$

is not valid. By above relations, it is equivalent to  $p_1 \equiv 1 \pmod{2}$ . Thus, in this case, (AS) over 2 is equivalent to two conditions:

$$\text{if } 2|\gamma, \gamma_2 = 1 \text{ and } 2|a_1, \text{ then } p_1 \equiv 1 \pmod{2} \quad (3.3.15)$$

and

$$\text{if } 2|\gamma, \gamma_2 = 1 \text{ and } 4|a_1, \text{ then } \delta - \gamma\mu^2 \not\equiv 0 \pmod{8a_1 b_1 c^2 / \gamma}. \quad (3.3.16)$$

Case  $\gamma_2 = 1$ ,  $2|\gamma$  and  $2|b_1$ . Then  $2 \leq \gamma^{(2)} = \gamma_b^{(2)}|b_1$ ,  $\gamma_a^{(2)} = 1$  and  $a_1 \equiv 1 \pmod{2}$ . By (3.2.50), (3.2.51) and (3.2.32), we get respectively

$$\mu \equiv 1 \pmod{2}, \quad (3.3.17)$$

$$\delta \equiv \gamma\mu^2 \pmod{4(b_1^{(2)}/\gamma_b^{(2)})}, \quad (3.3.18)$$

and then  $\delta \equiv \gamma \pmod{4}$ ,

$$p_1 - \mu q_1 \equiv 0 \pmod{2}. \quad (3.3.19)$$

Over 2, the conditions (3.2.35)—(3.2.40) give respectively

$$\tilde{b}_1 q_1 \equiv 0 \pmod{\gamma_b^{(2)}}, \quad (3.3.20)$$

$$q_1(p_1 - \mu q_1) \equiv 0 \pmod{2\gamma_b^{(2)}}, \quad (3.3.21)$$

$$p_1 q_1 \equiv q_1^2 \equiv 0 \pmod{\gamma_b^{(2)}}, \quad (3.3.22)$$

$$(\delta - \gamma\mu^2)q_1^2 \equiv 0 \pmod{4(\gamma_b^{(2)})^2(b_1^{(2)}/\gamma_b^{(2)})}, \quad (3.3.23)$$

$$\gamma p_1^2 + \delta q_1^2 \equiv 2\gamma\mu p_1 q_1 \pmod{8\gamma_b^{(2)}} \quad (3.3.24)$$

is not satisfied if  $4|b_1$ ,

$$\begin{cases} (b_1/\gamma_b)p_1 q_1 \equiv 0 \pmod{2\gamma_b^{(2)}} \\ \mu\gamma p_1^2 + \mu\delta q_1^2 \equiv 2\delta p_1 q_1 \pmod{8\gamma_b^{(2)}} \end{cases} \quad (3.3.25)$$

is not satisfied.

By (3.3.23), we have  $\delta q_1^2 \equiv \gamma\mu^2 q_1^2 \pmod{8\gamma_b^{(2)}}$ . It follows that (3.3.24) is equivalent to

$$p_1 - \mu q_1 \not\equiv 0 \pmod{4} \quad (3.3.26)$$

if  $4|b_1$ . Similarly (3.3.25) is equivalent to

$$\begin{cases} (b_1/\gamma_b)p_1 q_1 \equiv 0 \pmod{2\gamma_b^{(2)}} \\ p_1 - \mu q_1 \equiv 0 \pmod{4} \end{cases} \quad (3.3.27)$$

is not satisfied.

Assume that

$$(b_1/\gamma_b)p_1 q_1 \not\equiv 0 \pmod{2\gamma_b^{(2)}}. \quad (3.3.28)$$

By (3.3.22), we have  $p_1 q_1 \equiv 0 \pmod{\gamma_b^{(2)}}$ . Then (3.3.28) is equivalent to  $(b_1/\gamma_b)$  is odd and  $p_1 q_1 \equiv \gamma_b^{(2)} \pmod{2\gamma_b^{(2)}}$ . By (3.3.21), then  $q_1^2 \equiv \gamma_b^{(2)} \pmod{2\gamma_b^{(2)}}$ . Then  $\gamma_b^{(2)}$  is a square and  $4|b_1$ . By (3.3.26), then  $p_1 - \mu q_1 \not\equiv 0 \pmod{4}$ . Thus, (3.3.26) and (3.3.27) are equivalent to

$$p_1 - \mu q_1 \equiv 2 \pmod{4} \quad (3.3.29)$$

since  $p_1 \equiv q_1 \equiv 0 \pmod{2}$  by (3.3.17), (3.3.19) and (3.3.22). By (3.3.21), then  $q_1 \equiv 0 \pmod{\gamma_b^{(2)}}$ , and all other conditions of (AS) over 2 follow from here.

Thus, in this case, (AS) over 2 is equivalent to

$$\text{if } 2|\gamma, \gamma_2 = 1, \text{ and } 2|b_1, \text{ then } p_1 - \mu q_1 \not\equiv 0 \pmod{4} \text{ and } q_1 \equiv 0 \pmod{\gamma_b^{(2)}}. \quad (3.3.30)$$

Case a prime odd  $l|\gamma$  and  $l|a_1$ . Then  $l \leq \gamma^{(l)} = \gamma_a^{(l)}|a_1$ ,  $\gamma_b^{(l)} = 1$  and  $b_1 \not\equiv 0 \pmod{l}$ .

By (3.2.50), (3.2.51), (3.2.32) we have respectively

$$\mu \in (\mathbb{Z}/(a_1^{(l)}/\gamma_a^{(l)}))^*, \quad (3.3.31)$$

$$\delta \equiv \gamma\mu^2 \pmod{(a_1^{(l)}/\gamma_a^{(l)})}, \quad (3.3.32)$$

$$p_1 - \mu q_1 \equiv 0 \pmod{(a_1^{(l)}/\gamma_a^{(l)})}. \quad (3.3.33)$$

Conditions (AS) satisfy over  $l$  except (3.2.39) and (3.2.40) which give respectively

$$\begin{cases} \delta q_1^2 \equiv 0 \pmod{l} \\ \gamma p_1^2 + \delta q_1^2 \equiv 2\gamma\mu p_1 q_1 \pmod{(a_1^{(l)}/\gamma_a^{(l)})l} \end{cases} \quad (3.3.34)$$

is not satisfied if  $l^2|a_1$ ,

$$\begin{cases} \delta q_1^2 \equiv 0 \pmod{l} \\ p_1 q_1 \equiv 0 \pmod{l} \\ \mu\gamma p_1^2 + \mu\delta q_1^2 \equiv 2\delta p_1 q_1 \pmod{(a_1^{(l)}/\gamma_a^{(l)})l} \end{cases} \quad (3.3.35)$$

is not satisfied if  $l|(a_1^{(l)}/\gamma_a^{(l)})$ .

Taking square of (3.3.33), we obtain

$$\gamma p_1^2 + \gamma\mu^2 q_1^2 \equiv 2\gamma\mu p_1 q_1 \pmod{(a_1^{(l)}/\gamma_a^{(l)})l}. \quad (3.3.36)$$

This shows that (3.3.34) is equivalent to

$$\begin{cases} \delta q_1^2 \equiv 0 \pmod{l} \\ (\delta - \gamma\mu^2)q_1^2 \equiv 0 \pmod{(a_1^{(l)}/\gamma_a^{(l)})l} \end{cases} \quad (3.3.37)$$

is not satisfied if  $l^2|a_1$ . By (3.3.33), this is equivalent to

$$\begin{aligned} &\text{If odd prime } l|\gamma \text{ and } l^2|a_1, \text{ then } q_1 \not\equiv 0 \pmod{l} \text{ and} \\ &\text{either } \delta \not\equiv 0 \pmod{l} \text{ or } (\delta - \gamma\mu^2) \not\equiv 0 \pmod{(a_1^{(l)}/\gamma_a^{(l)})l}. \end{aligned} \quad (3.3.38)$$

If  $l|(a_1^{(l)}/\gamma_a^{(l)})$ , then  $l^2|a_1$  and (3.3.38) is valid. Then  $q_1 \not\equiv 0 \pmod{l}$ . By (3.3.31) and (3.3.33), then  $p_1 \not\equiv 0 \pmod{l}$ . Thus (3.3.35) follows from (3.3.38).

Finally we get that (AS) over  $l$  is equivalent to (3.3.38) in this case.

*Case a prime odd  $l \mid \gamma$  and  $l \mid b_1$ .* Then  $l \leq \gamma^{(l)} = \gamma_b^{(l)} \mid b_1$ ,  $\gamma_a^{(l)} = 1$  and  $a_1 \not\equiv 0 \pmod l$ .

By (3.2.50), (3.2.51), (3.2.32) we have respectively

$$\mu \in (\mathbb{Z}/(b_1^{(l)}/\gamma_b^{(l)}))^*, \quad (3.3.39)$$

$$\delta \equiv \gamma\mu^2 \pmod{(b_1^{(l)}/\gamma_b^{(l)})}, \quad (3.3.40)$$

$$\gamma p_1^2 - \delta q_1^2 = \pm 2(2/\gamma_2)(a_1/\gamma_a)\gamma_b c \quad (3.3.41)$$

Over  $l$ , the conditions (3.2.35)—(3.2.40) give respectively

$$\tilde{b}_1 q_1 \equiv 0 \pmod{\gamma_b^{(l)}}, \quad (3.3.42)$$

$$q_1(p_1 - \mu q_1) \equiv 0 \pmod{\gamma_b^{(l)}}, \quad (3.3.43)$$

$$p_1 q_1 \equiv q_1^2 \equiv 0 \pmod{\gamma_b^{(l)}}, \quad (3.3.44)$$

$$(\delta - \gamma\mu^2)q_1^2 \equiv 0 \pmod{(b_1^{(l)}/\gamma_b^{(l)})(\gamma_b^{(l)})^2}, \quad (3.3.45)$$

$$\begin{cases} (b_1^{(l)}/\gamma_b^{(l)})(\gamma p_1^2 + \delta q_1^2) \equiv 0 \pmod{\gamma_b^{(l)}l} \\ \gamma p_1^2 + \delta q_1^2 \equiv 2\gamma\mu p_1 q_1 \pmod{\gamma_b^{(l)}l} \end{cases} \quad (3.3.46)$$

is not satisfied if  $l^2 \mid b_1$ ,

$$\begin{cases} (b_1^{(l)}/\gamma_b^{(l)})(\gamma p_1^2 + \delta q_1^2) \equiv 0 \pmod{\gamma_b^{(l)}l} \\ \mu\gamma p_1^2 + \mu\delta q_1^2 \equiv 2\delta p_1 q_1 \pmod{\gamma_b^{(l)}l} \end{cases} \quad (3.3.47)$$

is not satisfied if  $l \mid (b_1^{(l)}/\gamma_b^{(l)})$ .

By (3.3.45), we have  $\delta q_1^2 \equiv \gamma\mu^2 q_1^2 \pmod{\gamma_b l}$ . Then (3.3.46) is equivalent to

$$\begin{cases} (b_1^{(l)}/\gamma_b^{(l)})(p_1^2 + \mu^2 q_1^2) \equiv 0 \pmod l \\ p_1 - \mu q_1 \equiv 0 \pmod l \end{cases} \quad (3.3.48)$$

is not satisfied if  $l^2 \mid b_1$ . Similarly (using also (3.3.39)) one can see that (3.3.47) is equivalent to

$$\begin{cases} (b_1^{(l)}/\gamma_b^{(l)})(p_1^2 + \mu^2 q_1^2) \equiv 0 \pmod l \\ p_1 - \mu q_1 \equiv 0 \pmod l \end{cases} \quad (3.3.49)$$

is not satisfied if  $l \mid (b_1^{(l)}/\gamma_b^{(l)})$ . Thus, (3.3.48) implies (3.3.49), and it is enough to satisfy (3.3.48).

Assume that  $p_1 - \mu q_1 \not\equiv 0 \pmod{l}$ . By (3.3.43) we get  $q_1 \equiv 0 \pmod{\gamma_b^{(l)}}$ , and all other conditions follow.

Assume that  $p_1 - \mu q_1 \equiv 0 \pmod{l}$ . Then  $p_1^2 + \mu^2 q_1^2 \equiv 2p_1^2 \pmod{l}$ , and (3.3.48) implies that  $p_1 \not\equiv 0 \pmod{l}$ . By (3.3.44),  $q_1 \equiv 0 \pmod{l}$ , and we get a contradiction. Since  $q_1 \equiv 0 \pmod{l}$ , the condition  $p_1 \not\equiv \mu q_1 \pmod{l}$  can be replaced by  $p_1 \not\equiv 0 \pmod{l}$ .

Thus, in this case, the condition (AS) over  $l$  is equivalent to two conditions

$$\text{If odd prime } l|\gamma \text{ and } l|b_1, \text{ then } q_1 \equiv 0 \pmod{\gamma_b^{(l)}}. \quad (3.3.50)$$

$$\text{If odd prime } l|\gamma \text{ and } l^2|b_1, \text{ then } p_1 \not\equiv 0 \pmod{l}. \quad (3.3.51)$$

Thus, we obtain

**Theorem 3.3.1.** *The singular condition (AS) is equivalent to*

$$\begin{aligned} &\text{if odd prime } l|\gamma \text{ and } l^2|a_1, \text{ then } q_1 \not\equiv 0 \pmod{l} \text{ and} \\ &\text{either } \delta \not\equiv 0 \pmod{l} \text{ or } (\delta - \gamma\mu^2) \not\equiv 0 \pmod{(a_1^{(l)}/\gamma_a^{(l)})l}; \\ &\text{if odd prime } l|\gamma \text{ and } l|b_1, \text{ then } q_1 \equiv 0 \pmod{\gamma_b^{(l)}}; \\ &\text{if odd prime } l|\gamma \text{ and } l^2|b_1, \text{ then } p_1 \not\equiv 0 \pmod{l}; \\ &\text{if } 2|\gamma, \gamma_2 = 1 \text{ and } 2|a_1, \text{ then } p_1 \equiv 1 \pmod{2}; \\ &\text{if } 2|\gamma, \gamma_2 = 1 \text{ and } 4|a_1, \text{ then } \delta - \gamma\mu^2 \not\equiv 0 \pmod{(8a_1b_1c^2/\gamma)}; \\ &\text{if } 2|\gamma, \gamma_2 = 1, \text{ and } 2|b_1, \text{ then } p_1 - \mu q_1 \not\equiv 0 \pmod{4} \text{ and } q_1 \equiv 0 \pmod{\gamma_b^{(2)}}; \\ &\text{if } 2|\gamma, \gamma_2 = 2 \text{ and } 2|b_1, \text{ then } p_1 \equiv 1 \pmod{2} \text{ and } q_1 \equiv 0 \pmod{\gamma^{(2)}/2}. \end{aligned} \quad (3.3.52)$$

*The singular condition (BS) is equivalent to*

$$\begin{aligned} &\text{if odd prime } l|\gamma \text{ and } l|a_1, \text{ then } q_1 \equiv 0 \pmod{\gamma_a^{(l)}}; \\ &\text{if odd prime } l|\gamma \text{ and } l^2|a_1, \text{ then } p_1 \not\equiv 0 \pmod{l}; \\ &\text{if odd prime } l|\gamma \text{ and } l^2|b_1, \text{ then } q_1 \not\equiv 0 \pmod{l} \text{ and} \\ &\text{either } \delta \not\equiv 0 \pmod{l} \text{ or } (\delta - \gamma\mu^2) \not\equiv 0 \pmod{(b_1^{(l)}/\gamma_b^{(l)})l}; \\ &\text{if } 2|\gamma, \gamma_2 = 1, \text{ and } 2|a_1, \text{ then } p_1 - \mu q_1 \not\equiv 0 \pmod{4} \text{ and } q_1 \equiv 0 \pmod{\gamma_a^{(2)}}; \\ &\text{if } 2|\gamma, \gamma_2 = 1 \text{ and } 2|b_1, \text{ then } p_1 \equiv 1 \pmod{2}; \\ &\text{if } 2|\gamma, \gamma_2 = 1 \text{ and } 4|b_1, \text{ then } \delta - \gamma\mu^2 \not\equiv 0 \pmod{(8a_1b_1c^2/\gamma)}; \\ &\text{if } 2|\gamma, \gamma_2 = 2 \text{ and } 2|a_1, \text{ then } p_1 \equiv 1 \pmod{2} \text{ and } q_1 \equiv 0 \pmod{\gamma^{(2)}/2}. \end{aligned} \quad (3.3.53)$$

Thus, we can finally rewrite the conditions (A) of  $a$ -series, and the conditions (B) of  $b$ -series in the very efficient form which makes all our results very effective.



**(A): The conditions of  $a$ -series:**

**(AG): The general conditions of  $a$ -series:**

$$p_1 - \mu q_1 \equiv 0 \pmod{(2/\gamma_2)(a_1/\gamma_a)c}, \quad (3.3.54)$$

$$p_1 - \mu q_1 \not\equiv 0 \pmod{(2/\gamma_2)(a_1/\gamma_a)cl}$$

for any prime  $l$  such that  $l^2|b_1$  and  $\text{g.c.d}(l, \gamma) = 1$ , (3.3.55)

$$l \nmid p_1 \text{ for any prime } l \text{ such that } l^2|a_1 \text{ and } \text{g.c.d}(l, \gamma) = 1. \quad (3.3.56)$$

**(AS) The singular conditions of  $a$ -series:**

*if odd prime  $l|\gamma$  and  $l^2|a_1$ , then  $q_1 \not\equiv 0 \pmod{l}$  and*  
*either  $\delta \not\equiv 0 \pmod{l}$  or  $(\delta - \gamma\mu^2) \not\equiv 0 \pmod{(a_1^{(l)}/\gamma_a^{(l)})l}$ ;*  
*if odd prime  $l|\gamma$  and  $l|b_1$ , then  $q_1 \equiv 0 \pmod{\gamma_b^{(l)}}$ ;*  
*if odd prime  $l|\gamma$  and  $l^2|b_1$ , then  $p_1 \not\equiv 0 \pmod{l}$ ;*  
*if  $2|\gamma$ ,  $\gamma_2 = 1$  and  $2|a_1$ , then  $p_1 \equiv 1 \pmod{2}$ ;*  
*if  $2|\gamma$ ,  $\gamma_2 = 1$  and  $4|a_1$ , then  $\delta - \gamma\mu^2 \not\equiv 0 \pmod{(8a_1b_1c^2/\gamma)}$ ;*  
*if  $2|\gamma$ ,  $\gamma_2 = 1$ , and  $2|b_1$ , then  $p_1 - \mu q_1 \not\equiv 0 \pmod{4}$  and  $q_1 \equiv 0 \pmod{\gamma_b^{(2)}}$ ;*  
*if  $2|\gamma$ ,  $\gamma_2 = 2$  and  $2|b_1$ , then  $p_1 \equiv 1 \pmod{2}$  and  $q_1 \equiv 0 \pmod{\gamma^{(2)}/2}$ .*  
(3.3.57)

**(B): The conditions of  $b$ -series:**

**(BG): The general conditions of  $b$ -series:**

$$p_1 - \mu q_1 \equiv 0 \pmod{(2/\gamma_2)(b_1/\gamma_b)c}, \quad (3.3.58)$$

$$p_1 - \mu q_1 \not\equiv 0 \pmod{(2/\gamma_2)(b_1/\gamma_b)cl}$$

for any prime  $l$  such that  $l^2|a_1$  and  $\text{g.c.d}(l, \gamma) = 1$ , (3.3.59)

$$l \nmid p_1 \text{ for any prime } l \text{ such that } l^2|b_1 \text{ and } \text{g.c.d}(l, \gamma) = 1. \quad (3.3.60)$$

**(BS) The singular conditions of  $b$ -series:**

*if odd prime  $l|\gamma$  and  $l|a_1$ , then  $q_1 \equiv 0 \pmod{\gamma_a^{(l)}}$ ;*  
*if odd prime  $l|\gamma$  and  $l^2|a_1$ , then  $p_1 \not\equiv 0 \pmod{l}$ ;*  
*if odd prime  $l|\gamma$  and  $l^2|b_1$ , then  $q_1 \not\equiv 0 \pmod{l}$  and*  
*either  $\delta \not\equiv 0 \pmod{l}$  or  $(\delta - \gamma\mu^2) \not\equiv 0 \pmod{(b_1^{(l)}/\gamma_b^{(l)})l}$ ;*  
*if  $2|\gamma$ ,  $\gamma_2 = 1$ , and  $2|a_1$ , then  $p_1 - \mu q_1 \not\equiv 0 \pmod{4}$  and  $q_1 \equiv 0 \pmod{\gamma_a^{(2)}}$ ;*  
*if  $2|\gamma$ ,  $\gamma_2 = 1$  and  $2|b_1$ , then  $p_1 \equiv 1 \pmod{2}$ ;*  
*if  $2|\gamma$ ,  $\gamma_2 = 1$  and  $4|b_1$ , then  $\delta - \gamma\mu^2 \not\equiv 0 \pmod{(8a_1b_1c^2/\gamma)}$ ;*  
*if  $2|\gamma$ ,  $\gamma_2 = 2$  and  $2|a_1$ , then  $p_1 \equiv 1 \pmod{2}$  and  $q_1 \equiv 0 \pmod{\gamma^{(2)}/2}$ .*  
(3.3.61)

We remind that here  $\gamma|2a_1b_1$  and

$$\mu \in (\mathbb{Z}/2a_1b_1c^2/\gamma)^*, \quad (3.3.62)$$

$$\delta \equiv \gamma\mu^2 \pmod{4a_1b_1c^2/\gamma}, \quad (3.3.63)$$

$$\gamma p_1^2 - \delta q_1^2 = \pm 2(2/\gamma_2)(a_1/\gamma_a)\gamma_b c \quad (3.3.64)$$

for the  $a$ -series, and

$$\gamma p_1^2 - \delta q_1^2 = \pm 2(2/\gamma_2)(b_1/\gamma_b)\gamma_a c \quad (3.3.65)$$

for the  $b$ -series.

#### 4. FINAL RESULTS FOR $\rho = 2$

Now we can formulate the main results which follow from Theorem 3.1.3 and the calculations above.

**Theorem 4.1.** *Let  $X$  be a K3 surface with  $\rho(X) = 2$ , and  $H$  a polarization of  $X$  of degree  $H^2 = 2rs$  where  $r, s \in \mathbb{N}$ . Assume that the Mukai vector  $(r, H, s)$  is primitive. Let  $Y$  be the moduli space of sheaves on  $X$  with the isotropic Mukai vector  $v = (r, H, s)$ . Let  $\tilde{H} = H/d$  be the corresponding primitive polarization, and  $\tilde{H} \cdot N(X) = \gamma\mathbb{Z}$ . We denote by  $\mu$  the invariant of the pair  $(N(X), \tilde{H})$  and use notations of Proposition 3.1.1.*

*We have  $Y \cong X$  if*

$$g.c.d(c, d\gamma) = 1$$

*and  $X$  belongs either to  $a$ -series or to  $b$ -series.*

*Here  $X$  belongs to  $a$ -series if for one of  $\epsilon = \pm 1$  the equation*

$$\gamma p_1^2 - \delta q_1^2 = \epsilon 2(2/\gamma_2)(a_1/\gamma_a)\gamma_b c. \quad (4.1)$$

*has an integral solution  $(p_1, q_1)$  satisfying the conditions (A) of  $a$ -series (3.3.54)–(3.3.57).*

*These solutions  $(p_1, q_1)$  of (4.1) give all solution  $(x, y)$  of Theorem 3.1.3 from  $a$ -series as associated solutions*

$$(x, y) = \pm \left( \frac{-2a_1b_1c}{\gamma} + \frac{\epsilon b_1\gamma_2\gamma_a p_1^2}{\gamma_b}, \frac{\epsilon b_1\gamma_2\gamma_a p_1 q_1}{\gamma_b} \right). \quad (4.2)$$

*Here  $X$  belongs to  $b$ -series if for one of signs  $\epsilon = \pm 1$  the equation*

$$\gamma p_1^2 - \delta q_1^2 = \epsilon 2(2/\gamma_2)(b_1/\gamma_b)\gamma_a c. \quad (4.3)$$

*has an integral solution  $(p_1, q_1)$  satisfying the conditions (B) of  $b$ -series (3.3.58)–(3.3.61).*

These solutions  $(p_1, q_1)$  of (4.3) give all solutions  $(x, y)$  of Theorem 3.1.3 of  $b$ -series as associated solutions

$$(x, y) = \pm \left( \frac{-2a_1 b_1 c}{\gamma} + \frac{\epsilon a_1 \gamma_2 \gamma_b p_1^2}{\gamma_a}, \frac{\epsilon a_1 \gamma_2 \gamma_b p_1 q_1}{\gamma_a} \right). \quad (4.4)$$

These conditions are necessary to have  $Y \cong X$  if  $X$  is a general K3 surface with  $\rho(X) = 2$ , i. e. the automorphism group of the transcendental periods  $(T(X), H^{2,0}(X))$  is  $\pm 1$ .

Now we want to interpret solutions  $(p_1, q_1)$  of Theorem 4.1 as elements of the Picard lattice  $N(X)$ .

Let  $(p_1, q_1)$  be a solution of Theorem 4.1 from  $a$ -series. Then

$$\gamma p_1^2 - \delta q_1^2 = \epsilon 2(2/\gamma_2)(a_1/\gamma_a)\gamma_b c. \quad (4.5)$$

By (3.3.54), we have

$$p_1 - \mu q_1 \equiv 0 \pmod{(2/\gamma_2)(a_1/\gamma_a)c}. \quad (4.6)$$

Let us put

$$t = (b_1/\gamma_b)c. \quad (4.7)$$

Then

$$tp_1 - \mu tq_1 \equiv 0 \pmod{2a_1 b_1 c^2/\gamma}. \quad (4.8)$$

and

$$\tilde{h}_1 = \frac{t(p_1 \tilde{H} + q_1 f(\tilde{H}))}{2a_1 b_1 c^2/\gamma} \in N(X). \quad (4.9)$$

We have

$$\tilde{h}_1^2 = \frac{t^2(\gamma p_1^2 - \delta q_1^2)}{2a_1 b_1 c^2/\gamma} = \epsilon 2b_1 c \quad (4.10)$$

and

$$\tilde{H} \cdot \tilde{h}_1 = \gamma(b_1/\gamma_b)cp_1 \equiv 0 \pmod{\gamma(b_1/\gamma_b)c} \text{ and } p_1 = \frac{\tilde{H} \cdot \tilde{h}_1}{\gamma(b_1/\gamma_b)c}. \quad (4.11)$$

Also

$$-f(\tilde{H}) \cdot \tilde{h}_1 = \frac{b_1 c \delta q_1}{\gamma_b} \text{ and } q_1 = -\frac{f(\tilde{H}) \cdot \tilde{h}_1}{\delta(b_1/\gamma_b)c}. \quad (4.12)$$

Here

$$-\gamma \delta = \det N(X). \quad (4.13)$$

Another calculation of  $p_1$  and  $q_1$  is as follows. We have

$$\tilde{h}_1 = \frac{u\tilde{H} + v f(\tilde{H})}{2a_1 b_1 c^2/\gamma} = \frac{p_1 \tilde{H} + q_1 f(\tilde{H})}{(2/\gamma_2)(a_1/\gamma_a)c} \quad (4.14)$$

where

$$u \equiv 0 \pmod{(b_1/\gamma_b)c}, \text{ and } p_1 = \frac{u}{(b_1/\gamma_b)c}, \quad (4.15)$$

$$v \equiv 0 \pmod{(b_1/\gamma_b)c}, \text{ and } q_1 = \frac{v}{(b_1/\gamma_b)c}. \quad (4.16)$$

We remind that here  $\mathbb{Z}f(\tilde{H})$  is the orthogonal complement to  $\tilde{H}$  in  $N(X)$ . Both these calculations of  $p_1$  and  $q_1$  are equivalent.

By construction, (3.3.54) is equivalent to  $\tilde{h}_1 \in N(X)$ , (3.3.55) is equivalent to  $\tilde{h}_1/l \notin N(X)$ , (3.3.56) is equivalent to  $\tilde{H} \cdot \tilde{h}_1 \not\equiv 0 \pmod{\gamma(b_1/\gamma_b)cl}$ .

Changing the letters  $a$  and  $b$  places we get the same results for the  $b$ -series.

Thus, we get

**Theorem 4.2.** *Let  $X$  be a K3 surface with  $\rho(X) = 2$ , and  $H$  a polarization of  $X$  of degree  $H^2 = 2rs$  where  $r, s \in \mathbb{N}$ . Assume that the Mukai vector  $(r, H, s)$  is primitive. Let  $Y$  be the moduli space of sheaves on  $X$  with the isotropic Mukai vector  $v = (r, H, s)$ . Let  $\tilde{H} = H/d$  be the corresponding primitive polarization, and  $\tilde{H} \cdot N(X) = \gamma\mathbb{Z}$ . We denote by  $\mu$  the invariant of the pair  $(N(X), \tilde{H})$  and use notations of Proposition 3.1.1.*

*We have  $Y \cong X$  if*

$$g.c.d(c, d\gamma) = 1,$$

*and at least for one  $\epsilon = \pm 1$  there exists  $\tilde{h}_1 \in N(X)$  which belongs to the  $a$ -series or to the  $b$ -series described below:*

*$\tilde{h}_1$  belongs to the  $a$ -series if*

$$\tilde{h}_1^2 = \epsilon 2b_1c \text{ and } \tilde{H} \cdot \tilde{h}_1 \equiv 0 \pmod{\gamma(b_1/\gamma_b)c}, \quad (4.17)$$

$$\tilde{H} \cdot \tilde{h}_1 \not\equiv 0 \pmod{\gamma(b_1/\gamma_b)cl_1}, \tilde{h}_1/l_2 \notin N(X) \quad (4.18)$$

*for any prime  $l_1$  such that  $l_1^2 | a_1$  and  $g.c.d(l_1, \gamma) = 1$ , and for any prime  $l_2$  such that  $l_2^2 | b_1$  and  $g.c.d(l_2, \gamma) = 1$ , and*

$$p_1 = \frac{\tilde{H} \cdot \tilde{h}_1}{\gamma(b_1/\gamma_b)c}, \quad q_1 = -\frac{f(\tilde{H}) \cdot \tilde{h}_1}{\delta(b_1/\gamma_b)c} \quad (4.19)$$

*satisfy the singular condition (AG) (conditions (3.3.57) mod  $\gamma$ ) of  $a$ -series.*

*$\tilde{h}_1$  belongs to the  $b$ -series if*

$$\tilde{h}_1^2 = \epsilon 2a_1c \text{ and } \tilde{H} \cdot \tilde{h}_1 \equiv 0 \pmod{\gamma(a_1/\gamma_a)c}, \quad (4.20)$$

$$\tilde{H} \cdot \tilde{h}_1 \not\equiv 0 \pmod{\gamma(a_1/\gamma_a)cl_1}, \tilde{h}_1/l_2 \notin N(X) \quad (4.21)$$

for any prime  $l_1$  such that  $l_1^2 | b_1$  and  $\text{g.c.d}(l_1, \gamma) = 1$ , and for any prime  $l_2$  such that  $l_2^2 | a_1$  and  $\text{g.c.d}(l_2, \gamma) = 1$ , and

$$p_1 = \frac{\tilde{H} \cdot \tilde{h}_1}{\gamma(a_1/\gamma_a)c}, \quad q_1 = -\frac{f(\tilde{H}) \cdot \tilde{h}_1}{\delta(a_1/\gamma_a)c} \quad (4.22)$$

satisfy the singular condition (BG) (conditions (3.3.61) mod  $\gamma$ ) of  $b$ -series.

These conditions are necessary to have  $Y \cong X$  if  $X$  is a general K3 surface with  $\rho(X) = 2$ , i. e. the automorphism group of the transcendental periods  $(T(X), H^{2,0}(X))$  is  $\pm 1$ .

*Important Remark 4.3.* Applying Theorem 3.1.3 and the formulae (4.2), (4.4) for the associated solution, we get the following formulae in terms of  $X$  for the canonical primitive *nef* element  $\tilde{h}$  of  $Y$  defined by  $(-a, 0, b) \bmod \mathbb{Z}v$  :

$$\tilde{h}' = \begin{cases} \frac{-\tilde{H}}{c} + \frac{\epsilon(\tilde{H} \cdot \tilde{h}_1)\tilde{h}_1}{b_1 c^2} & \text{if } \tilde{h}_1 \text{ is from } a\text{-series,} \\ \frac{-\tilde{H}}{c} + \frac{\epsilon(\tilde{H} \cdot \tilde{h}_1)\tilde{h}_1}{a_1 c^2} & \text{if } \tilde{h}_1 \text{ is from } b\text{-series} \end{cases} \quad (4.23)$$

belongs to  $N(X)$  and

$$(Y, \tilde{h}) \cong (X, \pm w(\tilde{h}')) \text{ for some } w \in W^{(-2)}(N(X)). \quad (4.24)$$

Specializing (by Lemma 2.2.1) the theorem 4.2, we get the following sufficient condition to have  $Y \cong X$  which is valid for  $X$  with any  $\rho(X)$ . This is one of the main results of the paper.

In Theorem 4.4 below, for  $\tilde{H} \in N$  we apply the same definitions and notations:  $f(\tilde{H})$ ,  $\delta$ ,  $\mu$ , as for  $\tilde{H} \in N = N(X)$  of Proposition 3.1.1.

**Theorem 4.4.** *Let  $X$  be a K3 surface and  $H$  a polarization of  $X$  of degree  $H^2 = 2rs$  where  $r, s \in \mathbb{N}$ . Assume that the Mukai vector  $(r, H, s)$  is primitive. Let  $Y$  be the moduli space of sheaves on  $X$  with the isotropic Mukai vector  $v = (r, H, s)$ . Let  $\tilde{H} = H/d$  be the corresponding primitive polarization.*

*We have  $Y \cong X$  if there exists  $\tilde{h}_1 \in N(X)$  such that  $\tilde{H}, \tilde{h}_1$  belong to a 2-dimensional primitive sublattice  $N \subset N(X)$  such that  $\tilde{H} \cdot N = \gamma\mathbb{Z}$ ,  $\gamma > 0$ , and*

$$\text{g.c.d}(c, d\gamma) = 1, \quad (4.25)$$

*moreover, for one of  $\epsilon = \pm 1$  the element  $\tilde{h}_1$  belongs to the  $a$ -series or to the  $b$ -series described below:*

*$\tilde{h}_1$  belongs to the  $a$ -series if*

$$\tilde{h}_1^2 = \epsilon 2b_1 c \text{ and } \tilde{H} \cdot \tilde{h}_1 \equiv 0 \pmod{\gamma(b_1/\gamma_b)c}, \quad (4.26)$$

$$\tilde{H} \cdot \tilde{h}_1 \not\equiv 0 \pmod{\gamma(b_1/\gamma_b)cl_1}, \quad \tilde{h}_1/l_2 \notin N(X) \quad (4.27)$$

for any prime  $l_1$  such that  $l_1^2|a_1$  and  $\text{g.c.d}(l_1, \gamma) = 1$ , and any prime  $l_2$  such that  $l_2^2|b_1$  and  $\text{g.c.d}(l_2, \gamma) = 1$ , and

$$p_1 = \frac{\tilde{H} \cdot \tilde{h}_1}{\gamma(b_1/\gamma_b)c}, \quad q_1 = -\frac{f(\tilde{H}) \cdot \tilde{h}_1}{\delta(b_1/\gamma_b)c} \quad (4.28)$$

satisfy the singular condition (AS) of  $a$ -series:

- if odd prime  $l|\gamma$  and  $l^2|a_1$ , then  $q_1 \not\equiv 0 \pmod{l}$  and either  $\delta \not\equiv 0 \pmod{l}$  or  $(\delta - \gamma\mu^2) \not\equiv 0 \pmod{(a_1^{(l)}/\gamma_a^{(l)})l}$ ;
- if odd prime  $l|\gamma$  and  $l|b_1$ , then  $q_1 \equiv 0 \pmod{\gamma_b^{(l)}}$ ;
- if odd prime  $l|\gamma$  and  $l^2|b_1$ , then  $p_1 \not\equiv 0 \pmod{l}$ ;
- if  $2|\gamma$ ,  $\gamma_2 = 1$  and  $2|a_1$ , then  $p_1 \equiv 1 \pmod{2}$ ;
- if  $2|\gamma$ ,  $\gamma_2 = 1$  and  $4|a_1$ , then  $\delta - \gamma\mu^2 \not\equiv 0 \pmod{(8a_1b_1c^2/\gamma)}$ ;
- if  $2|\gamma$ ,  $\gamma_2 = 1$ , and  $2|b_1$ , then  $p_1 - \mu q_1 \not\equiv 0 \pmod{4}$  and  $q_1 \equiv 0 \pmod{\gamma_b^{(2)}}$ ;
- if  $2|\gamma$ ,  $\gamma_2 = 2$  and  $2|b_1$ , then  $p_1 \equiv 1 \pmod{2}$  and  $q_1 \equiv 0 \pmod{\gamma^{(2)}/2}$ .

$\tilde{h}_1$  belongs to the  $b$ -series if

$$\tilde{h}_1^2 = \epsilon 2a_1c \text{ and } \tilde{H} \cdot \tilde{h}_1 \equiv 0 \pmod{\gamma(a_1/\gamma_a)c}, \quad (4.29)$$

$$\tilde{H} \cdot \tilde{h}_1 \not\equiv 0 \pmod{\gamma(a_1/\gamma_a)cl_1}, \quad \tilde{h}_1/l_2 \notin N(X) \quad (4.30)$$

for any prime  $l_1$  such that  $l_1^2|b_1$  and  $\text{g.c.d}(l_1, \gamma) = 1$  and any prime  $l_2$  such that  $l_2^2|a_1$  and  $\text{g.c.d}(l_2, \gamma) = 1$ , and

$$p_1 = \frac{\tilde{H} \cdot \tilde{h}_1}{\gamma(a_1/\gamma_a)c}, \quad q_1 = -\frac{f(\tilde{H}) \cdot \tilde{h}_1}{\delta(a_1/\gamma_a)c} \quad (4.31)$$

satisfy the singular condition (BS) of  $b$ -series:

- if odd prime  $l|\gamma$  and  $l|a_1$ , then  $q_1 \equiv 0 \pmod{\gamma_a^{(l)}}$ ;
- if odd prime  $l|\gamma$  and  $l^2|a_1$ , then  $p_1 \not\equiv 0 \pmod{l}$ ;
- if odd prime  $l|\gamma$  and  $l^2|b_1$ , then  $q_1 \not\equiv 0 \pmod{l}$  and either  $\delta \not\equiv 0 \pmod{l}$  or  $(\delta - \gamma\mu^2) \not\equiv 0 \pmod{(b_1^{(l)}/\gamma_b^{(l)})l}$ ;
- if  $2|\gamma$ ,  $\gamma_2 = 1$ , and  $2|a_1$ , then  $p_1 - \mu q_1 \not\equiv 0 \pmod{4}$  and  $q_1 \equiv 0 \pmod{\gamma_a^{(2)}}$ ;
- if  $2|\gamma$ ,  $\gamma_2 = 1$  and  $2|b_1$ , then  $p_1 \equiv 1 \pmod{2}$ ;
- if  $2|\gamma$ ,  $\gamma_2 = 1$  and  $4|b_1$ , then  $\delta - \gamma\mu^2 \not\equiv 0 \pmod{(8a_1b_1c^2/\gamma)}$ ;
- if  $2|\gamma$ ,  $\gamma_2 = 2$  and  $2|a_1$ , then  $p_1 \equiv 1 \pmod{2}$  and  $q_1 \equiv 0 \pmod{\gamma^{(2)}/2}$ .

Moreover, one has formulae (4.23) and (4.24) in terms of  $X$  for the canonical primitive nef element  $\tilde{h}$  of  $Y$  defined by  $(-a, 0, b) \bmod \mathbb{Z}v$ .

These conditions are necessary to have  $Y \cong X$  if  $\rho(X) \leq 2$  and  $X$  is a general K3 surface with its Picard lattice, i. e. the automorphism group of the transcendental periods  $(T(X), H^{2,0}(X))$  is  $\pm 1$ .

See Sect. 6 about the cases  $\gamma = 1$  and  $\gamma = 2$ .

## 5. DIVISORIAL CONDITIONS ON MODULI OF $(X, H)$ WHICH IMPLY $Y \cong X$ AND $\gamma(\tilde{H}) = \gamma$

Using notations above, assuming  $\text{g.c.d}(c, d\gamma) = 1$  for

$$\bar{\mu} = \{\mu, -\mu\} \subset (\mathbb{Z}/(2a_1b_1c^2/\gamma))^*, \quad \epsilon = \pm 1$$

we denote by

$$\mathcal{D}(r, s, d, \gamma; A)_{\epsilon}^{\bar{\mu}} \quad (5.1)$$

the set of all  $\delta \in \mathbb{N}$  such that  $\delta \equiv \gamma\mu^2 \bmod 4a_1b_1c^2/\gamma$  and the equation  $\gamma p_1^2 - \delta q_1^2 = \epsilon 2(2/\gamma_2)(a_1/\gamma_a)\gamma_b c$  has an integral solution  $(p_1, q_1)$  satisfying the condition (A) (3.3.54)—(3.3.57) of the  $a$ -series. Similarly, we define

$$\mathcal{D}(r, s, d, \gamma; B)_{\epsilon}^{\bar{\mu}} \quad (5.2)$$

the  $b$ -series changing  $a$  and  $b$  places (see the equation (3.3.65) and conditions (B) (3.3.58)—(3.3.61)).

We denote

$$\begin{aligned} \mathcal{D}(r, s, d, \gamma)^{\bar{\mu}} = & \left( \bigcup_{\epsilon \in \{-1, 1\}} \mathcal{D}(r, s, d, \gamma; A)_{\epsilon}^{\bar{\mu}} \right) \\ & \bigcup \left( \bigcup_{\epsilon \in \{-1, 1\}} \mathcal{D}(r, s, d, \gamma; B)_{\epsilon}^{\bar{\mu}} \right). \end{aligned} \quad (5.3)$$

By Theorem 4.1, the set  $\mathcal{D}(r, s, d, \gamma)^{\bar{\mu}}$  describes all possible pairs  $H \in N(X)$  of general polarized K3 surfaces  $(X, H)$  with  $\text{rk } N(X) = 2$ , the primitive polarization  $\tilde{H} = H/d \in N(X)$ , the invariant  $\gamma(\tilde{H}) = \gamma$  (i. e.  $\tilde{H} \cdot N(X) = \gamma\mathbb{Z}$ ) and the invariant  $\pm\mu$  for  $\tilde{H} \in N(X)$ , such that  $Y \cong X$ . By general results of [N1] and [N2] the pair  $\tilde{H} \in N(X)$  defines the irreducible 18-dimensional moduli of such pairs  $(X, \tilde{H})$ , i. e. a (irreducible) divisorial condition on 19-dimensional moduli of polarized K3 surfaces  $(X, H)$  which implies that  $\gamma(\tilde{H}) = \gamma$  and  $Y \cong X$ . Thus, we can interpret our results as follows.

**Theorem 5.1.** *The set*

$$\mathcal{D}(r, s, d, \gamma) = \{(\bar{\mu}, \delta) \mid \{\mu, -\mu\} \subset (\mathbb{Z}/(2a_1b_1c^2/\gamma))^*, \delta \in \mathcal{D}(r, s, d, \gamma)^{\bar{\mu}}\} \quad (5.4)$$

*describes all irreducible divisorial conditions on moduli of polarized K3 surfaces  $(X, H)$  with  $H^2 = 2rs$  and the primitive polarization  $\tilde{H} = H/d$  (here  $d^2 \mid rs$ ), which imply  $Y \cong X$  for any  $X$ , and  $\tilde{H} \cdot N(X) = \gamma\mathbb{Z}$  for a general  $X$ .*

*We have (see (5.3))*

$$\begin{aligned} \mathcal{D}(r, s, d, \gamma)^{\bar{\mu}} = & \left( \bigcup_{\epsilon \in \{-1, 1\}} \mathcal{D}(r, s, d, \gamma; A)_{\epsilon}^{\bar{\mu}} \right) \\ & \bigcup \left( \bigcup_{\epsilon \in \{-1, 1\}} \mathcal{D}(r, s, d, \gamma; B)_{\epsilon}^{\bar{\mu}} \right). \end{aligned} \quad (5.5)$$

*where each set  $\mathcal{D}(r, s, d, \gamma; A)_{\epsilon}^{\bar{\mu}}$  and  $\mathcal{D}(r, s, d, \gamma; B)_{\epsilon}^{\bar{\mu}}$  is infinite if it is not empty.*

*Proof.* We need to prove the last statement only. Assume that  $\mathcal{D}(r, s, d, \gamma; A)_{\epsilon}^{\bar{\mu}}$  is not empty. Thus, there exist integral  $(p_0, q_0)$  such that

$$\begin{cases} \frac{\gamma p_0^2 - \epsilon 2(2/\gamma_2)(a_1/\gamma_a)\gamma_b c}{q_0^2} \equiv \gamma \mu^2 \pmod{\frac{4a_1b_1c^2}{\gamma}} \\ \gamma p_0^2 - \epsilon 2(2/\gamma_2)(a_1/\gamma_a)\gamma_b c > 0 \\ (p_0, q_0) \text{ satisfies } (A) \end{cases} \quad (5.6)$$

Then

$$\delta_0 = \frac{\gamma p_0^2 - \epsilon 2(2/\gamma_2)(a_1/\gamma_a)\gamma_b c}{q_0^2} \in \mathcal{D}(r, s, d, \gamma; A)_{\epsilon}^{\bar{\mu}}. \quad (5.7)$$

The (5.6) is equivalent to

$$\begin{cases} \gamma p_0^2 - \epsilon 2(2/\gamma_2)(a_1/\gamma_a)\gamma_b c \equiv \gamma \mu^2 q_0^2 \pmod{4a_1b_1c^2 q_0^2/\gamma} \\ \gamma p_0^2 - \epsilon 2(2/\gamma_2)(a_1/\gamma_a)\gamma_b c > 0 \\ (p_0, q_0) \text{ satisfies } (A) \end{cases} \quad (5.8)$$

Clearly,  $(p, q_0)$  where

$$p \equiv p_0 \pmod{8a_1b_1c^2 q_0^2}, \text{ and } \gamma p_0^2 - \epsilon 2(2/\gamma_2)(a_1/\gamma_a)\gamma_b c > 0 \quad (5.9)$$

also satisfies (5.8) and defines

$$\delta = \frac{\gamma p_0^2 - \epsilon 2(2/\gamma_2)(a_1/\gamma_a)\gamma_b c}{q_0^2} \in \mathcal{D}(r, s, d, \gamma; A)_{\epsilon}^{\bar{\mu}}. \quad (5.10)$$

Obviously, their number is infinite. This proves the statement.

The key question is:



**Problem 5.2.** When  $\mathcal{D}(r, s, d, \gamma; A)_\epsilon^\mu$  and  $\mathcal{D}(r, s, d, \gamma; B)_\epsilon^\mu$  are non-empty?

We hope to consider this question in further publications on the subject. It was shown in [N4] (see also [MN1], [MN2]) that at least one of these sets is not empty if  $d = 1$  and  $\gamma = 1$ . Theorem 4.1 and exactly the same considerations as in [N4] show that it is also valid for  $\gamma = 1$  and any  $d$  because singular conditions (AS) and (BS) satisfy if  $\gamma = 1$ . Thus we have

**Theorem 5.3.** At least for one of  $\bar{\mu}, \epsilon$  one of sets  $\mathcal{D}(r, s, d, \gamma = 1; A)_\epsilon^\mu$  or  $\mathcal{D}(r, s, d, \gamma = 1; B)_\epsilon^\mu$  is not empty.

In particular, for any primitive isotropic Mukai vector  $(r, H, s)$  the set of divisorial conditions on moduli of  $X$  which imply that  $Y \cong X$  and  $\gamma = 1$  is not empty and is then infinite.

We hope to consider Problem 5.2 for other  $\gamma$  in further publications on the subject.

## 6. EXAMPLES OF $\gamma = 1$ AND $\gamma = 2$

As concrete examples of results of Sect. 4, we consider cases of  $\gamma = 1$  and  $\gamma = 2$ .

When  $\gamma = 1$ , then singular conditions (AS) and (BS) are obviously valid, and we obtain especially simple results. We formulate only the analogy of Theorem 4.4.

**Theorem 6.1.** Let  $X$  be a K3 surface and  $H$  a polarization of  $X$  of degree  $H^2 = 2rs$  where  $r, s \in \mathbb{N}$ . Assume that the Mukai vector  $(r, H, s)$  is primitive. Let  $Y$  be the moduli space of sheaves on  $X$  with the isotropic Mukai vector  $v = (r, H, s)$ . Let  $\tilde{H} = H/d$  be the corresponding primitive polarization.

We have  $Y \cong X$  if there exists  $\tilde{h}_1 \in N(X)$  such that  $\tilde{H}, \tilde{h}_1$  belong to a 2-dimensional primitive sublattice  $N \subset N(X)$  such that

$$\tilde{H} \cdot N = \mathbb{Z} \tag{6.1}$$

(i. e.  $\gamma = 1$ ), moreover, for one of  $\epsilon = \pm 1$  the element  $\tilde{h}_1$  belongs to the  $a$ -series or to the  $b$ -series described below:

$\tilde{h}_1$  belongs to the  $a$ -series if

$$\tilde{h}_1^2 = \epsilon 2b_1c \text{ and } \tilde{H} \cdot \tilde{h}_1 \equiv 0 \pmod{b_1c}, \tag{6.2}$$

$$\tilde{H} \cdot \tilde{h}_1 \not\equiv 0 \pmod{b_1cl_1}, \tilde{h}_1/l_2 \notin N(X) \tag{6.3}$$

for any prime  $l_1$  such that  $l_1^2 | a_1$ , and any prime  $l_2$  such that  $l_2^2 | b_1$ .

$\tilde{h}_1$  belongs to the  $b$ -series if

$$\tilde{h}_1^2 = \epsilon 2a_1c \text{ and } \tilde{H} \cdot \tilde{h}_1 \equiv 0 \pmod{a_1c}, \tag{6.4}$$

$$\tilde{H} \cdot \tilde{h}_1 \not\equiv 0 \pmod{a_1cl_1}, \tilde{h}_1/l_2 \notin N(X) \tag{6.5}$$

for any prime  $l_1$  such that  $l_1^2 | b_1$  and any prime  $l_2$  such that  $l_2^2 | a_1$ .

Moreover, one has formulae (4.23) and (4.24) in terms of  $X$  for the canonical primitive nef element  $\tilde{h}$  of  $Y$  defined by  $(-a, 0, b) \pmod{\mathbb{Z}v}$ .

These conditions are necessary to have  $Y \cong X$  and  $\tilde{H} \cdot N(X) = \mathbb{Z}$  if  $\rho(X) \leq 2$  and  $X$  is a general K3 surface with its Picard lattice, i. e. the automorphism group of the transcendental periods  $(T(X), H^{2,0}(X))$  is  $\pm 1$ .

This generalizes results of [MN1], [MN2] and [N4] where the condition  $H \cdot N = \mathbb{Z}$  had been imposed (i. e.  $d = \gamma = 1$ ).

Now let us assume that  $\gamma = 2$ . By Theorem 4.4, we obtain three cases which all together cover all possibilities for  $\gamma = 2$ .

When  $\gamma = 2$  and  $a_1 \equiv b_1 \equiv 1 \pmod{2}$ , then  $\gamma_2 = 2$ , and singular conditions (AS) and (BS) satisfy. Theorem 4.4 gives then

**Theorem 6.2.** *Let  $X$  be a K3 surface and  $H$  a polarization of  $X$  of degree  $H^2 = 2rs$  where  $r, s \in \mathbb{N}$ . Assume that the Mukai vector  $(r, H, s)$  is primitive. Let  $Y$  be the moduli space of sheaves on  $X$  with the isotropic Mukai vector  $v = (r, H, s)$ . Let  $\tilde{H} = H/d$  be the corresponding primitive polarization. Assume that*

$$g.c.d(2, c) = 1 \tag{6.6}$$

and

$$a_1 \equiv b_1 \equiv 1 \pmod{2}. \tag{6.7}$$

We have  $Y \cong X$  if there exists  $\tilde{h}_1 \in N(X)$  such that  $\tilde{H}, \tilde{h}_1$  belong to a 2-dimensional primitive sublattice  $N \subset N(X)$  such that

$$\tilde{H} \cdot N = 2\mathbb{Z} \tag{6.8}$$

(then  $\gamma = 2$ ,  $\gamma_a = \gamma_b = 1$  and  $\gamma_2 = 2$ ), moreover, for one of  $\epsilon = \pm 1$  the element  $\tilde{h}_1$  belongs to the  $a$ -series or to the  $b$ -series described below:

$\tilde{h}_1$  belongs to the  $a$ -series if

$$\tilde{h}_1^2 = \epsilon 2b_1c \text{ and } \tilde{H} \cdot \tilde{h}_1 \equiv 0 \pmod{2b_1c}, \tag{6.9}$$

$$\tilde{H} \cdot \tilde{h}_1 \not\equiv 0 \pmod{2b_1cl_1}, \tilde{h}_1/l_2 \notin N(X) \tag{6.10}$$

for any prime  $l_1$  such that  $l_1^2 | a_1$ , and any prime  $l_2$  such that  $l_2^2 | b_1$ .

$\tilde{h}_1$  belongs to the  $b$ -series if

$$\tilde{h}_1^2 = \epsilon 2a_1c \text{ and } \tilde{H} \cdot \tilde{h}_1 \equiv 0 \pmod{2a_1c}, \tag{6.11}$$

$$\tilde{H} \cdot \tilde{h}_1 \not\equiv 0 \pmod{2a_1cl_1}, \tilde{h}_1/l_2 \notin N(X) \tag{6.12}$$

for any prime  $l_1$  such that  $l_1^2 | b_1$  and any prime  $l_2$  such that  $l_2^2 | a_1$ .

Moreover, one has formulae (4.23) and (4.24) in terms of  $X$  for the canonical primitive nef element  $\tilde{h}$  of  $Y$  defined by  $(-a, 0, b) \bmod \mathbb{Z}v$ .

These conditions are necessary (for odd  $c$ ,  $a_1$  and  $b_1$ ) to have  $Y \cong X$  and  $\tilde{H} \cdot N(X) = 2\mathbb{Z}$  if  $\rho(X) \leq 2$  and  $X$  is a general K3 surface with its Picard lattice, i. e. the automorphism group of the transcendental periods  $(T(X), H^{2,0}(X))$  is  $\pm 1$ .

In [MN2] the primitive isotropic Mukai vector  $(c, H, c)$  where  $H^2 = 2c^2$  had been considered. Then  $a = b = 1$ ,  $d = 1$ ,  $a_1 = b_1 = 1$  and  $\gamma|2$ . The case  $\gamma = 1$  had been described in [MN2]. Theorem 6.2 describes the remaining case  $\gamma = 2$  and then  $c$  is odd which was not considered in [MN2].

Now assume that  $\gamma = 2$  and  $2|a_1$ . Then  $\gamma_2 = 1$ ,  $\gamma_a = 2$  and  $\gamma_b = 1$ . The singular condition (AS) gives then (6.18) and (6.19) below. The singular condition (BS) gives (6.22) and (6.23) below. Thus, Theorem 4.4 implies the following.

**Theorem 6.3.** *Let  $X$  be a K3 surface and  $H$  a polarization of  $X$  of degree  $H^2 = 2rs$  where  $r, s \in \mathbb{N}$ . Assume that the Mukai vector  $(r, H, s)$  is primitive. Let  $Y$  be the moduli space of sheaves on  $X$  with the isotropic Mukai vector  $v = (r, H, s)$ . Let  $\tilde{H} = H/d$  be the corresponding primitive polarization. Assume that*

$$g.c.d(2, c) = 1 \tag{6.13}$$

and

$$a_1 \equiv 0 \pmod{2}, \quad b_1 \equiv 1 \pmod{2}. \tag{6.14}$$

We have  $Y \cong X$  if there exists  $\tilde{h}_1 \in N(X)$  such that  $\tilde{H}, \tilde{h}_1$  belong to a 2-dimensional primitive sublattice  $N \subset N(X)$  such that

$$\tilde{H} \cdot N = 2\mathbb{Z} \tag{6.15}$$

(then  $\gamma = 2$ ,  $\gamma_a = 2$ ,  $\gamma_b = 1$  and  $\gamma_2 = 1$ ), moreover, for one of  $\epsilon = \pm 1$  the element  $\tilde{h}_1$  belongs to the  $a$ -series or to the  $b$ -series described below:

$\tilde{h}_1$  belongs to the  $a$ -series if

$$\tilde{h}_1^2 = \epsilon 2b_1c \text{ and } \tilde{H} \cdot \tilde{h}_1 \equiv 0 \pmod{2b_1c}, \tag{6.16}$$

$$\tilde{H} \cdot \tilde{h}_1 \not\equiv 0 \pmod{2b_1cl_1}, \quad \tilde{h}_1/l_2 \notin N(X) \tag{6.17}$$

for any prime  $l_1$  such that  $l_1^2|a_1$  and  $g.c.d(l_1, 2) = 1$ , and any prime  $l_2$  such that  $l_2^2|b_1$ ; moreover (singular conditions),

$$\tilde{H} \cdot \tilde{h}_1 \not\equiv 0 \pmod{4b_1c} \tag{6.18}$$

and

$$\delta \not\equiv 2\mu^2 \pmod{4a_1} \text{ if } 4|a_1. \tag{6.19}$$

$\tilde{h}_1$  belongs to the  $b$ -series if

$$\tilde{h}_1^2 = \epsilon 2a_1c \text{ and } \tilde{H} \cdot \tilde{h}_1 \equiv 0 \pmod{a_1c}, \quad (6.20)$$

$$\tilde{H} \cdot \tilde{h}_1 \not\equiv 0 \pmod{a_1cl_1}, \quad \tilde{h}_1/l_2 \notin N(X) \quad (6.21)$$

for any prime  $l_1$  such that  $l_1^2 | b_1$  and any prime  $l_2$  such that  $l_2^2 | a_1$  and  $\text{g.c.d.}(l_2, 2) = 1$ ; moreover (singular conditions),

$$\tilde{H} \cdot \tilde{h}_1 \equiv 0 \pmod{2a_1c} \quad (6.22)$$

and

$$\tilde{h}_1/2 \notin N(X). \quad (6.23)$$

Moreover, one has formulae (4.23) and (4.24) in terms of  $X$  for the canonical primitive nef element  $\tilde{h}$  of  $Y$  defined by  $(-a, 0, b) \pmod{\mathbb{Z}v}$ .

These conditions are necessary (for odd  $c$ , even  $a_1$  and odd  $b_1$ ) to have  $Y \cong X$  and  $\tilde{H} \cdot N(X) = 2\mathbb{Z}$  if  $\rho(X) \leq 2$  and  $X$  is a general K3 surface with its Picard lattice, i. e. the automorphism group of the transcendental periods  $(T(X), H^{2,0}(X))$  is  $\pm 1$ .

Changing  $a$  and  $b$  places, we get from Theorem 4.4 the remaining case.

**Theorem 6.4.** *Let  $X$  be a K3 surface and  $H$  a polarization of  $X$  of degree  $H^2 = 2rs$  where  $r, s \in \mathbb{N}$ . Assume that the Mukai vector  $(r, H, s)$  is primitive. Let  $Y$  be the moduli space of sheaves on  $X$  with the isotropic Mukai vector  $v = (r, H, s)$ . Let  $\tilde{H} = H/d$  be the corresponding primitive polarization. Assume that*

$$\text{g.c.d.}(2, c) = 1 \quad (6.24)$$

and

$$a_1 \equiv 1 \pmod{2}, \quad b_1 \equiv 0 \pmod{2}. \quad (6.25)$$

We have  $Y \cong X$  if there exists  $\tilde{h}_1 \in N(X)$  such that  $\tilde{H}, \tilde{h}_1$  belong to a 2-dimensional primitive sublattice  $N \subset N(X)$  such that

$$\tilde{H} \cdot N = 2\mathbb{Z} \quad (6.26)$$

(then  $\gamma = 2$ ,  $\gamma_a = 1$ ,  $\gamma_b = 2$  and  $\gamma_2 = 1$ ), moreover, for one of  $\epsilon = \pm 1$  the element  $\tilde{h}_1$  belongs to the  $a$ -series or to the  $b$ -series described below:

$\tilde{h}_1$  belongs to the  $a$ -series if

$$\tilde{h}_1^2 = \epsilon 2b_1c \text{ and } \tilde{H} \cdot \tilde{h}_1 \equiv 0 \pmod{b_1c}, \quad (6.27)$$

$$\tilde{H} \cdot \tilde{h}_1 \not\equiv 0 \pmod{b_1cl_1}, \quad \tilde{h}_1/l_2 \notin N(X) \quad (6.28)$$

for any prime  $l_1$  such that  $l_1^2 | a_1$  and any prime  $l_2$  such that  $l_2^2 | b_1$  and  $\text{g.c.d.}(l_2, 2) = 1$ ; moreover (singular conditions),

$$\tilde{H} \cdot \tilde{h}_1 \equiv 0 \pmod{2b_1c} \quad (6.29)$$

and

$$\tilde{h}_1/2 \notin N(X). \quad (6.30)$$

$\tilde{h}_1$  belongs to the  $b$ -series if

$$\tilde{h}_1^2 = \epsilon 2a_1c \text{ and } \tilde{H} \cdot \tilde{h}_1 \equiv 0 \pmod{2a_1c}, \quad (6.31)$$

$$\tilde{H} \cdot \tilde{h}_1 \not\equiv 0 \pmod{2a_1cl_1}, \tilde{h}_1/l_2 \notin N(X) \quad (6.32)$$

for any prime  $l_1$  such that  $l_1^2 | b_1$  and  $\text{g.c.d.}(l_1, 2) = 1$ , and any prime  $l_2$  such that  $l_2^2 | a_1$ ; moreover (singular conditions),

$$\tilde{H} \cdot \tilde{h}_1 \not\equiv 0 \pmod{4a_1c} \quad (6.33)$$

and

$$\delta \not\equiv 2\mu^2 \pmod{4b_1} \text{ if } 4 | b_1. \quad (6.34)$$

Moreover, one has formulae (4.23) and (4.24) in terms of  $X$  for the canonical primitive nef element  $\tilde{h}$  of  $Y$  defined by  $(-a, 0, b) \pmod{\mathbb{Z}v}$ .

These conditions are necessary (for odd  $c$ , odd  $a_1$  and even  $b_1$ ) to have  $Y \cong X$  and  $\tilde{H} \cdot N(X) = 2\mathbb{Z}$  if  $\rho(X) \leq 2$  and  $X$  is a general K3 surface with its Picard lattice, i. e. the automorphism group of the transcendental periods  $(T(X), H^{2,0}(X))$  is  $\pm 1$ .

Theorems 6.2 — 6.4 cover all types of a primitive isotropic Mukai vector when it is in principle possible to have  $Y \cong X$  and  $\gamma = 2$ .

Using results of Sect. 4, one can write down similar very concrete and effective results for any  $\gamma$ .

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